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On third order integrable vector Hamiltonian equations

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ABSTRACT

A complete list of third order vector Hamiltonian equations with the Hamiltonian operator D_x having an infinite series of higher conservation laws is presented. A new vector integrable equation on the sphere is found.

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1. Introduction

Consider evolution vector equations of the following form

$$\boldsymbol{u}_t = f_0\boldsymbol{u} + f_1\boldsymbol{u}_1 + f_2\boldsymbol{u}_2 + f_3\boldsymbol{u}_3. \tag{1.1}$$

Here $\boldsymbol{u} \in \mathbb{R}^N$ is unknown vector-function of two variables t and x, $\boldsymbol{u}_n$ is its nth x-derivative, $\boldsymbol{u}_0 \equiv \boldsymbol{u}$, and $\boldsymbol{u}_t$ is the t-derivative of $\boldsymbol{u}$. The coefficients f_i, $i = 0, \ldots, 3$ are some scalar valued functions depending on several scalar products $(\boldsymbol{u}_i, \boldsymbol{u}_k) \equiv u_{[i,k]}$, $0 \leqslant i \leqslant k \leqslant 3$. Under additional restriction $u_{0,0} = 1$ integrable equation (1.1) has been investigated in [1]. A classification of some special classes of integrable equation (1.1) can be found in [2,3]. Any complete classification of integrable equation (1.1) is not done yet.

In this paper we consider Eq. (1.1) Hamiltonian with respect to the Gardner–Faddeev–Zakharov Poisson bracket. Any such equation has the form

$$\boldsymbol{u}_t = D_x \frac{\delta}{\delta \boldsymbol{u}} H(u_{[0,0]}, u_{[0,1]}, u_{[1,1]}), \tag{1.2}$$

where D_x is the total derivative with respect to x and $\frac{\delta}{\delta \boldsymbol{u}}$ is the vectorial variational derivative

$$\frac{\delta}{\delta \boldsymbol{u}} = \sum_{0 \leqslant i \leqslant k < \infty} \left[(-D_x)^i \boldsymbol{u}_k \frac{\partial}{\partial u_{[i,k]}} + (-D_x)^k \boldsymbol{u}_i \frac{\partial}{\partial u_{[i,k]}} \right].$$

For the first order Hamiltonians we have

$$\frac{\delta}{\delta \boldsymbol{u}} H = 2\frac{\partial H}{\partial u_{[0,0]}} \boldsymbol{u} + \frac{\partial H}{\partial u_{[0,1]}} \boldsymbol{u}_x - D_x\left(\frac{\partial H}{\partial u_{[0,1]}} \boldsymbol{u}\right) - 2\, D_x\left(\frac{\partial H}{\partial u_{[1,1]}} \boldsymbol{u}_x\right).$$

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We find a complete (up to canonical transformations) list of Hamiltonians H such that the corresponding Eq. (1.2) possesses infinitely many higher local conservation laws of the form

$$\frac{\partial \rho}{\partial t} = \frac{\partial \sigma}{\partial x}.$$

Here ρ and σ are scalar-valued functions depending on a finite number of the scalar products $u_{[i,k]}$. We call such equations *integrable*. A transformation of t, x, $\boldsymbol{u}$ that do not change the form of the Hamiltonian operator D_x is said to be the *canonical*. In particular, the transformations

$$x \to k_1 x, \qquad t \to k_2 t, \qquad \boldsymbol{u} \to k_3 \boldsymbol{u}, \tag{1.3}$$

and the Galilei transform

$$x \to x + k_4 t, \tag{1.4}$$

with k_i being constants, are canonical ones.

It is easy to verify that for any c_1 Hamiltonians H and $\tilde{H} = H + c_1 u_{[0,0]}$ are equivalent with respect to the Galilei transform. If $\tilde{H} = H + D_x f(u_{[0,0]}, u_{[0,1]}, \ldots) + c_2$, then $\frac{\delta H}{\delta \boldsymbol{u}} = \frac{\delta \tilde{H}}{\delta \boldsymbol{u}}$ and the corresponding Hamiltonian Eqs. (1.2) coincide. If $\tilde{H} - H = D_x(f) + c_1 u_{[0,0]} + c_2$, we call H and $\tilde{H}$ *equivalent*.

Let us formulate the main classification result.

Theorem 1. *If the nonlinear* equation (1.2) *admits infinitely many local conservation laws, then the Hamiltonian up to scalings* (1.3) *is equivalent to one of the following Hamiltonians:*

$$H_1 = \frac{\boldsymbol{u}_x^2}{(\boldsymbol{u}^2 + c)^3}, \tag{1.5}$$

$$H_2 = \frac{\boldsymbol{u}_x^2}{|\boldsymbol{u}|^3} + q\big(\boldsymbol{u}^2, (\boldsymbol{u}, \boldsymbol{u}_x)\big), \tag{1.6}$$

$$H_3 = |\boldsymbol{u}_x|^{1/2}, \tag{1.7}$$

$$H_4 = (\boldsymbol{u}_x^2 + 2\,(\boldsymbol{u}, \boldsymbol{u}_x) + \boldsymbol{u}^2)^{1/4}, \tag{1.8}$$

$$H_5 = (\boldsymbol{u}_x^2 + 2\,\boldsymbol{u}^2(\boldsymbol{u}, \boldsymbol{u}_x) + |\boldsymbol{u}|^6)^{1/4}, \tag{1.9}$$

$$H_6 = 2\,(\boldsymbol{u}_x^2 - \boldsymbol{u}^{-2}(\boldsymbol{u}, \boldsymbol{u}_x)^2)^{1/4} + q\big(\boldsymbol{u}^2, (\boldsymbol{u}, \boldsymbol{u}_x)\big), \tag{1.10}$$

where c is a constant, q is an arbitrary function of two variables.

The paper is organized as follows. In Section 2 we elaborate necessary integrability conditions for equations of the form (1.1). For scalar evolution equations similar conditions have been proposed in [4–9]. In the vector case they have been introduced in [1]. A reduced proof of the Theorem 1 is presented in Section 3. In Section 4 we discuss Eqs. (1.5)–(1.10). In particular, we consider the scalar limit $n = 1$ for these equations, prove the integrability of Eqs. (1.5), (1.7), (1.8), and explain the existence of arbitrary functions in Hamiltonians (1.6) and (1.10).

2. Integrability conditions

To find all integrable Hamiltonian equation (1.2) we use the componentless version of the symmetry approach for classification of the integrable equations [4]. This approach is based on the notion of the so-called *canonical conservation laws*

$$D_t \rho_i = D_x \theta_i, \quad i = -1, 0, 1, \ldots. \tag{2.1}$$

In the scalar equations this notion has been introduced in [8,10]. In this case ρ_i and θ_i depend on unknown function u and finite number of its x-derivatives and can be found recursively. It was shown in [7] that if the equation has infinitely many non-trivial local conservation laws, then for even $i = 2j$ the canonical densities are trivial: $\rho_{2j} = D_x \varphi_{2j}$, or the same,

$$\frac{\delta \rho_{2j}}{\delta u} = 0, \qquad j \geqslant 0. \tag{2.2}$$

In [1,2] this approach has been generalized to the vector equations. In this case ρ_i and θ_i are functions of a finite number of scalar products $u_{[i,j]}$. According to [1], if a vector equation has an infinite series of local conservation laws, then (2.2) holds.

The simplest algebraic algorithm for calculating the canonical densities ρ_i was presented in [5]. Afterward this algorithm was clarified in [6,11] and modified to the vector case in [1]. To deduce a recursive formula for the canonical densities for equations of the form (1.1) we rewrite the equation in the operator form

$$A\boldsymbol{u} = 0, \qquad A = -D_t + f_3 D_x^3 + f_2 D_x^2 + f_1 D_x + f_0.$$

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