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Deformed cohomologies of symmetry pseudo-groups and coverings of differential equations

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The work establishes a relation between deformed cohomologies of symmetry pseudo-groups and coverings of differential equations. Examples include the potential Khokhlov–Zabolotskaya equation and the Boyer–Finley equation.
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1. Introduction

Deformed cohomologies were introduced in [1–7] as a tool in the theory of analytic functions of several complex variables in geometry of Poisson manifolds, and in the Morse theory for smooth multi-valued functionals. Then they were applied to different problems of symplectic geometry and algebraic topology, see e.g. [8–11]. The objective of the present paper is to establish a relation between the deformed cohomologies of symmetry pseudo-groups of partial differential equations and their coverings.

Differential coverings (or Wahlquist–Estabrook prolongation structures, [12], or zero-curvature representations, [13], or integrable extensions, [14], etc.) are of great importance in geometry of PDEs. The theory of coverings is a natural framework for dealing with inverse scattering constructions for soliton equations, Bäcklund transformations, recursion operators, nonlocal symmetries and nonlocal conservation laws, Darboux transformations, and deformations of nonlinear PDEs, [15–17]. A number of techniques have been devised to handle the problem of recognizing whether a given differential equation has a covering, [12,18–28]. In [29], examples of coverings of PDEs with three independent variables were found by means of Élie Cartan's method of equivalence, [30–35]. This idea was developed in [36–38]. In [39] we propose an approach to the covering problem based on the technique of contact integrable extensions (CIEs) of the structure equations of the symmetry pseudo-groups, which is a generalization of the definition of integrable extension from [14, §6] for the

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case of more than two independent variables. Then in [40–42] the method of CIEs was applied to find coverings, Bäcklund transformations, and recursion operators for a number of PDEs.

All the above-mentioned techniques use ad-hoc assumptions about the possible form of covering equations. In this paper we propose a method which allows one to get rid of such assumptions and to formulate the solution of the covering problem in the internal terms of the PDE under the study. Our approach is based on the observation that a non-trivial deformed 2-cocycle of the symmetry pseudo-group of a PDE defines an integrable extension of this pseudo-group. Then a covering for the PDE may be obtained by integration of the extension equation in accordance with Lie's third inverse fundamental theorem in Cartan's form, [30,32,43,44].

In this paper we consider two equations: the potential Khokhlov–Zabolotskaya equation (or Lin–Reissner–Tsien equation), [45,46],

$$u_{yy} = u_{tx} + u_x\, u_{xx}, \tag{1}$$

and the Boyer–Finley equation, [47],

$$u_{tx} = \mathrm{e}^{u_y} u_{yy}. \tag{2}$$

We show that symmetry pseudo-groups of both equations have non-trivial deformed second cohomologies. The integrable extensions that correspond to cocycles from these cohomology groups define known coverings of Eqs. (1) and (2).

2. Preliminaries

2.1. Coverings of PDEs

All considerations in this paper are local. The presentation in this subsection closely follows to [48,49]. Let $\pi\colon \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, $\pi\colon (x^1,\dots,x^n,u^1,\dots,u^m) \mapsto (x^1,\dots,x^n)$ be a trivial bundle, and $J^\infty(\pi)$ be the bundle of its jets of the infinite order. The local coordinates on $J^\infty(\pi)$ are $(x^i, u^\alpha, u^\alpha_I)$, where $I = (i_1,\dots,i_n)$ is a multi-index, and for every local section $f\colon \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^m$ of π the corresponding infinite jet $j_\infty(f)$ is a section $j_\infty(f)\colon \mathbb{R}^n \to J^\infty(\pi)$ such that $u^\alpha_I(j_\infty(f)) = \frac{\partial^{\#I} f^\alpha}{\partial x^I} = \frac{\partial^{i_1+\dots+i_n} f^\alpha}{(\partial x^1)^{i_1}\dots(\partial x^n)^{i_n}}$. We put $u^\alpha = u^\alpha_{(0,\dots,0)}$. Also, in the case of $n = 3, m = 1$ we denote $x^1 = t, x^2 = x, x^3 = y$, and $u^1_{(i,j,k)} = u_{t\dots tx\dots xy\dots y}$ with i times t, j times x, and k times y.

The vector fields

$$D_{x^k} = \frac{\partial}{\partial x^k} + \sum_{\#I\ge 0}\sum_{\alpha=1}^{m} u^\alpha_{I+1_k}\,\frac{\partial}{\partial u^\alpha_I}, \qquad k \in \{1,\dots,n\},$$

with $(i_1,\dots,i_k,\dots,i_n) + 1_k = (i_1,\dots,i_k+1,\dots,i_n)$ are referred to as *total derivatives*. They commute everywhere on $J^\infty(\pi)$: $[D_{x^i}, D_{x^j}] = 0$.

A system of PDEs $F_r(x^i, u^\alpha_I) = 0$, $\#I \le s$, $r \in \{1,\dots,R\}$, of the order $s \ge 1$ with $R \ge 1$ defines the submanifold $\mathcal{E} = \{(x^i, u^\alpha_I) \in J^\infty(\pi) \mid D_K(F_r(x^i, u^\alpha_I)) = 0,\ \#K \ge 0\}$ in $J^\infty(\pi)$.

Denote $\mathcal{W} = \mathbb{R}^\infty$ with coordinates w^s, $s \in \mathbb{N}\cup\{0\}$. Locally, an (infinite-dimensional) *differential covering* of $\mathcal{E}$ is a trivial bundle $\tau\colon J^\infty(\pi)\times\mathcal{W} \to J^\infty(\pi)$ equipped with the *extended total derivatives*

$$\tilde{D}_{x^k} = D_{x^k} + \sum_{s=0}^{\infty} T^s_k(x^i, u^\alpha_I, w^j)\,\frac{\partial}{\partial w^s} \tag{3}$$

such that $[\tilde{D}_{x^i}, \tilde{D}_{x^j}] = 0$ for all $i \ne j$ whenever $(x^i, u^\alpha_I) \in \mathcal{E}$. For the partial derivatives of w^s which are defined as $w^s_{x^k} = \tilde{D}_{x^k}(w^s)$ we have the system of *covering equations*

$$w^s_{x^k} = T^s_k(x^i, u^\alpha_I, w^j).$$

This over-determined system of PDEs is compatible whenever $(x^i, u^\alpha_I) \in \mathcal{E}$.

Dually the covering with extended total derivatives (3) is defined by the integrable ideal of the *Wahlquist–Estabrook forms*

$$\varpi^s = dw^s - T^s_k(x^i, u^\alpha_I, w^j)\,dx^k.$$

2.2. Cartan's structure theory of Lie pseudo-groups

Let M be a manifold of dimension n. A *local diffeomorphism* on M is a diffeomorphism $\Phi\colon \mathcal{U} \to \hat{\mathcal{U}}$ of two open subsets of M. A *pseudo-group* $\mathfrak{G}$ on M is a collection of local diffeomorphisms of M, which is closed under composition whenever the latter

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