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Periodic boundary conditions for KdV–Burgers equation on an interval

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ABSTRACT

For the KdV–Burgers equation on a finite interval the development of a regular profile starting from a constant one under a periodic perturbation on the boundary is studied. The equation describes a medium which is both dissipative and dispersive. For an appropriate combination of dispersion and dissipation the asymptotic profile looks like a periodical chain of shock fronts with a decreasing amplitude (similarly to the Burgers equation case). But due to dispersion each such front is followed by increasing oscillation leading to the next shock—like the ninth wave in rough seas. The development of such a profile is preceded by an initial shock of a constant height.

1. Introduction

The Kortveg–de Vries–Burgers equation

$$u_t(x,t) = \varepsilon^2 u_{xx}(x,t) - 2u(x,t)u_x(x,t) + \lambda u_{xxx}(x,t) \tag{1}$$

is related to the viscous and dispersive medium. The viscosity dampens oscillations (which are inherent to the dispersion) except for stationary solutions which are invariant for some subalgebra of the full symmetry algebra of the equation. On the whole line only bounded solutions are usually taken into account since only they have a physical meaning. It is not the case for a finite interval as an unbounded solution may still remain bounded within an interval. Thus we obtain some new effects.

We consider the initial value–boundary problem (IVBP) for the KdV–Burgers equation on a finite interval:

$$u(x,0) = f(x),\ u(a,t) = l(t),\ u_x(a,t) = L(t),\ u_x(b,t) = R(t),\ x \in [a,b]. \tag{2}$$

The case of the boundary conditions $u(a,t) = B + A\sin(\omega t),\quad u_x(b,t) = 0$ and related asymptotics are of a special interest here.

Some of our results are similar to those of [1] and Dubrovin et al. [2,3] that deals with a formation of dispersive shocks in a class of Hamiltonian dispersive regularizations of the quasi-linear transport equation. For the KdV–Burgers equation the shocks resulting in breaks (and preceded by a multi-oscillation) develop for some IVBPs; some other IVBPs lead to a monotonic convergence to an invariant solutions. One more possibility for the asymptotics is a class of periodic 'ninth-wave' profile solutions. Such profiles (though for traveling waves on a line) are known in nonlinear acoustics [4,5]; they form in media where nonlinearity dominates over dispersion, diffraction and absorption. The present paper is a continuation of our previous work [6].

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2. Traveling wave solutions

The KdV–Burgers equation (1) has three classical symmetries:

1. u_t—t-translation;
2. u_x—x-translation;
3. $1/2 - tu_x$—the Galilean symmetry.

It is convenient to interpret them as the flows commuting with one generated by (1). Thus if $u_0(x, t)$ is a solution to (1) and S is a symmetry then the solution of the problem

$$u_\tau = S, \quad u|_{\tau=0} = u_0$$

is also a solution to (1). This is the so called process of generation solutions by a symmetry. In this way

1. t-translation generates $u_0(x, t + \tau)$;
2. x-translation generates $u_0(x + \tau, t)$;
3. the Galilean symmetry generates $u_0(x - \tau t, t) + \frac{\tau}{2}$.

A traveling wave solution is of the form $u = u(x - Vt)$. It can be generated from a solution to (1) of the form $u = y(x)$ by the Galilean symmetry at $\tau = V$.

Hence to obtain the traveling wave solution to the Kortveg–de Vries–Burgers equation we start with its time-independent solutions. They satisfy the ordinary differential equation

$$\lambda y''' + \varepsilon^2 y'' - 2yy' = 0, \quad y' = \frac{dy}{dx},$$

whose order may be reduced:

$$\lambda y'' + \varepsilon^2 y' - y^2 + C = 0. \tag{3}$$

Since this equation is autonomous, its order may be reduced still further. Put $y' = p(y)$, then $y'' = p(y)\frac{d}{dy}p(y) = pp'$; it follows

$$\lambda pp' + \varepsilon^2 p = y^2 - C = 0. \tag{4}$$

Changing variables $p = -q/\varepsilon^2,\ y = z \cdot \lambda/\varepsilon^4$ we obtain

$$qq' - q = \frac{\lambda^2}{\varepsilon^8} z^2 - C.$$

The latter equation is a particular case of the well studied second kind Abel equation $qq' - q = g(z)$. For a special case $g(z) = Az^2 - 9/(625A)$ its general solution may be given in an implicit (parametric) form, [7], in terms the classical Weierstrass's elliptic function $\mathcal{P}$:

$$z = 5a\left(\tau\mathcal{P} \mp \frac{1}{2}\right), \quad z = a\tau^2(\tau\sqrt{\pm(4\mathcal{P}^3 - 1)} + 2\mathcal{P}), \quad A = \pm\frac{2}{125}a^{-1}.$$

Here τ is a parameter,

$$\tau = \int \frac{d\mathcal{P}}{\sqrt{\pm(4\mathcal{P}^3 - 1)}} - C_2, \mathcal{P} = \mathcal{P}(\tau + C_2, 0, 1).$$

Happily, few explicit solutions may be found. The formula $g(z) = Az^2 - 9/(625A) = (25Az - 3)(25Az + 3)/(625A)$ prompts still another change of variables: $s = \sqrt{25Az \pm 3},\ R = 25Aq$. The equation now reads

$$RR_s - 2sR = \frac{2}{25}(s^5 \mp 6s^3),$$

and one can easily find its polynomial solutions:

$$R(s) = \pm\frac{\sqrt{6}}{15}s^3 + \frac{2}{5}s^2. \tag{5}$$

Returning these solutions to the initial variables and in the case $A = \lambda^2/\varepsilon^8$ we obtain

$$y(x) = \frac{3\varepsilon^4 \tanh^2(\frac{\varepsilon^2 x}{10\lambda})}{50\lambda} - \frac{3\varepsilon^4 \tanh(\frac{\varepsilon^2 x}{10\lambda})}{25\lambda} - \frac{3\varepsilon^4}{50\lambda}. \tag{6}$$