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Quantum cohomology and quantum hydrodynamics from supersymmetric quiver gauge theories

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ABSTRACT

We study the connection between $\mathcal{N} = 2$ supersymmetric gauge theories, quantum cohomology and quantum integrable systems of hydrodynamic type. We consider gauge theories on ALE spaces of A and D-type and discuss how they describe the quantum cohomology of the corresponding Nakajima's quiver varieties. We also discuss how the exact evaluation of local BPS observables in the gauge theory can be used to calculate the spectrum of quantum Hamiltonians of spin Calogero integrable systems and spin Intermediate Long Wave hydrodynamics. This is explicitly obtained by a Bethe Ansatz Equation provided by the quiver gauge theory in terms of its adjacency matrix.

1. Introduction and discussion

The study of BPS correlators in $\mathcal{N} = 2$ supersymmetric gauge theories reveals to be a rich source of results in various branches of modern mathematical physics, ranging from classical [1–3] and quantum [4] integrable systems to topological invariants [5–7]. In this paper we use exact results in supersymmetric gauge theories to highlight new connections between quantum cohomology of algebraic varieties [8] and quantum integrable systems, focusing on Nakajima's quiver varieties [9]. These have a manifold interest, since they host representations of infinite dimensional Lie algebrae of Kac–Moody type; moreover, they naturally describe moduli spaces of Yang–Mills instantons on ALE spaces [10] and are linked to free [11–13] and interacting [14–17] two-dimensional conformal field theories. This reflects in the special nature of the corresponding quantum integrable systems, which reveal to be of *hydrodynamical* type, namely admitting an infinite complete set of quantum Hamiltonians in involution. The vev of BPS local operators in the gauge theory on the Ω-background captures the spectrum of these Hamiltonians [18–20]. The prototypical example for six-dimensional gauge theories is the quantum gl_N Intermediate Long Wave system [21,22] associated to the equivariant quantum cohomology of the ADHM quiver variety [19,23,24,20]. In the four dimensional limit this reduces to the correspondence between instanton counting and Benjamin–Ono quantum system discussed in [25,26]. Our considerations are based on the intriguing interplay taking place between the description of supersymmetric vacua of D-brane systems and quiver representation theory [27]. The superstring background hosting the D-brane system fixes the quiver type, while the D-branes dimensions select the abelian category where the quiver representation is realized. We analyse in detail the D1–D5 system on ALE spaces and show that it provides a description of the equivariant quantum cohomology of the associated Nakajima's quiver varieties and links it to quantum spin Calogero and spin Intermediate Long Wave integrable systems. More precisely, we study the supersymmetric gauged linear sigma model (GLSM) on S^2 which describes the low-energy D1-branes dynamics by calculating its exact partition

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function. In the *Higgs phase*, this model flows in the infrared to a non-linear sigma model with target space the Nakajima's quiver variety, naturally describing its equivariant quantum cohomology. An equivalent description of the gauged linear sigma model can be obtained in the *Coulomb phase*, giving rise to a Landau–Ginzburg model whose twisted superpotential is the Yang–Yang function [28] of a quantum integrable system which we identify with spin gl_N Intermediate Long Wave system [29]. The Bethe Ansatz Equation are provided by the general quiver gauge theory in terms of its adjacency matrix which reduces to the Cartan matrix if $\epsilon_1 = \epsilon_2$. Once the BAE are formulated in these terms, they naturally extend to the affine ADE general case.

Thus the equivariant quantum cohomology/quantum integrable system correspondence is realized as an incarnation of mirror symmetry for *non-abelian* GLSMs.

The gauge theories we consider are obtained as low energy theories of D-branes configurations. The generic D-brane set up we consider is realized in the ten dimensional target space $\left(\widetilde{\mathbb{C}^2/\Gamma}\right) \times \mathcal{O}_{\mathbb{P}^1}(-2) \times \mathbb{C}$ where Γ is the ADE discrete group defining the ALE space as the quotient $\mathbb{C}^2/\Gamma$, $\left(\widetilde{\mathbb{C}^2/\Gamma}\right)$ is its minimal resolution and $\mathcal{O}_{\mathbb{P}^1}(-2)$ is the total space of the canonical bundle over $\mathbb{P}^1$. We place N D5-branes on $\left(\widetilde{\mathbb{C}^2/\Gamma}\right) \times \mathbb{P}^1$ and k D1-branes on $\mathbb{P}^1$ and consider the resulting quiver GLSM. It corresponds to the affine quiver $\hat{\Gamma}$, where the nodes are representing the GLSM input data corresponding to the dynamics on the common $\mathbb{P}^1$. Notice that, if Γ is trivial, then the resulting quiver gauge theory is the ADHM quiver. The resulting affine quiver has indeed a natural interpretation from the string theory point of view in that it keeps into account all the different low energy open string sectors.

In Section 2 we review the S^2 partition function and explain the quantum cohomology/quantum hydrodynamics mirror symmetry in a general set-up. We then focus on the ADHM quiver whose Higgs phase is described in Section 3 along with the description of its associated quantum cohomology problem in the non linear sigma model, while its Coulomb phase is described in Section 4 along with its quantum hydrodynamical associated model. The generalization to ALE quiver gauge theories is explained in Section 5, where we consider A_{p-1} and D_p quivers and the respective mirror phases. In particular in Section 5.3 we discuss the Bethe Ansatz equations for quantum spin gl_N ILW arising from the LG model and the connection with quantum spin Calogero model. Let us finally present some open issues.

- D1–D5 systems are naturally related to Donaldson–Thomas (DT) invariants. In particular, for the ADHM quiver, our results directly link to the ones obtained by Diaconescu [30] for non abelian local DT invariants on $\mathbb{C}^2 \times \mathbb{P}^1$. Therefore it would be useful to explore the relevance of our results for the computation of non abelian DT invariants on $\left(\widetilde{\mathbb{C}^2/\Gamma}\right) \times \mathbb{P}^1$.
- We did not consider E_n quivers, although it would be interesting to analyse this class of models too.
- The extension of our computations to the presence of gauge theory defects – as surface operators – along the way of [31] could pave the way to further applications of supersymmetric gauge theories to the quantization of integrable systems. For example, this could provide explicit expressions for the quantum ILW eigenfunctions and the corresponding quantum Hamiltonians.
- In the context of the AGT correspondence it was shown [25] that the basis of Virasoro descendants reproducing instanton counting on $\mathbb{C}^2$ has the special property of diagonalizing the Benjamin–Ono quantum Hamiltonians. Our results point towards an analogous connection between spin gl_N Benjamin–Ono quantum Hamiltonians and parafermionic $\mathcal{W}_N$-algebras that would be interesting to further explore.
- Finally, it would be interesting to understand how far the above correspondence goes in using quiver gauge theories to describe the quantization of integrable models of hydrodynamical type.

2. Gauged linear sigma models on S^2: generalities

Since in this paper we will be working with supersymmetric $\mathcal{N} = (2, 2)$ gauge theories on S^2, in this section we review the main points concerning localization on an euclidean two-sphere of radius r along the lines of [32,33], to which we refer for further details. After briefly reviewing the computation of the partition function Z_{S^2} for these theories, we will discuss how Z_{S^2} is related to Givental's approach to genus zero Gromov–Witten theory for Kähler manifolds when we consider the Higgs branch of the theory. On the other hand, the Coulomb branch has strong connections to quantum integrable systems via the so-called Bethe/Gauge correspondence; in the last subsection we will discuss the meaning of the partition function in that context.

2.1. $\mathcal{N} = (2, 2)$ gauge theories on S^2

The two-sphere S^2 is a conformally flat space; it does not admit Killing spinors, but it admits four complex conformal Killing spinors, which realize the $\mathfrak{osp}(2|2, \mathbb{C})$ superconformal algebra on S^2. We take as $\mathcal{N} = (2, 2)$ supersymmetry algebra on S^2 the subalgebra $\mathfrak{su}(2|1) \subset \mathfrak{osp}(2|2, \mathbb{C})$, realized by two out of the four conformal Killing spinors, which does not contain conformal nor superconformal transformations; its bosonic subalgebra $\mathfrak{su}(2) \oplus \mathfrak{u}(1)_R \subset \mathfrak{su}(2|1)$ generates the isometries of S^2 and an abelian vector R-symmetry, which is now part of the algebra and not an outer automorphism of it.

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