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Meromorphic Higgs bundles and related geometries

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ABSTRACT

The present note is mostly a survey on the generalised Hitchin integrable system and moduli spaces of meromorphic G-Higgs bundles. We also fill minor gaps in the existing literature, outline a calculation of the infinitesimal period map and review some related geometries.

1. Introduction

1.1. Integrable systems and complex geometry

Many moduli spaces arising in complex-algebraic or analytic geometry carry a symplectic or Poisson structure. The spaces considered in this survey are no exception. Let G be a simple complex Lie group, X a compact Riemann surface with canonical bundle $K_X = \Omega^1_X$ and D a sufficiently positive effective divisor on X. Our exposition is built around the study of meromorphic, i.e., $K_X(D)$-valued, G-Higgs bundles on X (Definition 2.1.) and their coarse moduli spaces $\mathbf{Higgs}_{G,D}$. These spaces come with the additional structure of an algebraic completely integrable Hamiltonian system (ACIHS), known as the generalised (or ramified) Hitchin system.

Completely integrable Hamiltonian systems have long been an object of interest for both mathematicians and physicists. The last thirty years have brought significant advances in the study of their algebraic (and holomorphic) counterparts. This was stimulated by the development of new methods in abelian and non-abelian Hodge theory, complex dynamics and holomorphic symplectic geometry, Yang–Mills and Seiberg–Witten theories, and of course, the quest for understanding mirror symmetry in its various incarnations.

The key difference between real and algebraic (or holomorphic) integrable systems is that abelian varieties (and complex tori) have moduli. Hence, after the removal of singular fibres, the structure morphism of the ACIHS is a C^∞ torus fibration, which usually fails to be holomorphically locally trivial. It is thus important to understand the corresponding period map, or, less ambitiously, the differential of the latter.

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1.2. Contents of the paper

We begin with a discussion of the moduli spaces $\mathbf{Higgs}_{G,D}$ (Section 2) and their Poisson geometry (Section 3). Then in Section 4 we discuss cameral covers and the Hitchin map. Most of the results in these introductory sections are standard and based on [1–4]. There are, however, a number of well-known (and used) extensions of results of Bottacin and Markman, for which we have not been able to locate a proper reference. For these we have included partial proofs, wherever appropriate.

One of our goals in this note is to outline a calculation of the infinitesimal period map of the generalised (ramified) Hitchin system. This is done in Section 5, and we refer to [5] for more details. In short, our main result in Section 5 is that the Balduzzi–Pantev formula [6,7] holds along the maximal rank symplectic leaves of the generalised Hitchin system.

Admittedly, the ramified Hitchin system may seem very special, but we recall the folklore statement that all known ACIHS arise as special case of Hitchin's. Some well-known examples are geodesic flows on ellipsoids (Jacobi–Moser–Mumford system), KP elliptic solitons, Calogero–Moser and elliptic Sklyanin systems. While some of these systems arise as complexifications of real CIHS, in general such complexifications do not give rise to ACIHS, since real Liouville tori need not "complexify well". We direct the interested reader to the wonderful surveys [3] and [4] for a detailed discussion and examples.

Apart from Higgs bundles, cameral covers and Prym varieties, there are several other geometric structures related to the space $\mathbf{Higgs}_{G,D}$: special Kähler geometry, several flavours of Hodge theory, tt^*-geometry and Frobenius-like structures, to name a few. We devote our final section Section 6 to a very brief literature review and discussion of some of these structures.

1.3. Conventions and notation

In Sections 2–4 we alternate between the holomorphic and the algebraic viewpoint and emphasise the differences, whenever deemed important. For the proof of the main theorem in Section 5 we work in the holomorphic category. We fix the following two types of ingredients:

(1) Geometric data:
- a smooth, compact, connected Riemann surface X of genus $g \geq 0$
- a divisor $D \geq 0$ on X, with $K_X(D)^2$ very ample

(2) Lie-theoretic data:
- a simple complex Lie group G
- Cartan and Borel subgroups $T \subset B \subset G$.

We denote by Z or $Z(G)$ the centre of G. The twist of the canonical bundle of X by $\mathcal{O}_X(D)$ will be denoted by $L := K_X(D)$. We shall also use the following – mostly standard – Lie-theoretic notation. The Lie algebras of the Cartan and Borel subgroups will be denoted, respectively, as $\mathfrak{t} \subset \mathfrak{b} \subset \mathfrak{g}$, while $\mathcal{R}^+ \subset \mathcal{R} \subset \mathfrak{t}^\vee$ will denote the (positive) roots. We let $W = N_G(T)/T$ stand for the abstract Weyl group, which will be identified with its embeddings in $GL(\mathfrak{t}^\vee)$ and $GL(\mathfrak{t})$. Finally, let $l = \operatorname{rk} \mathfrak{g} = \dim \mathfrak{t}$ be the rank of G, and d_i $(1 \leq i \leq l)$ the degrees of (any choice of) basic G-invariant polynomials on $\mathfrak{g}$. For some of the calculations we will also use a fixed choice of generators $\{I_i\}$ of $\mathbb{C}[\mathfrak{g}]^G$. We also fix an Ad-invariant symmetric bilinear form Tr on $\mathfrak{g}$.

To these data one can associate two (closely related) families of abelian torsors parametrised by the *Hitchin base* $\mathcal{B} = H^0(X, \mathfrak{t} \otimes_{\mathbb{C}} L/W) \simeq H^0(X, \bigoplus_i L^{d_i})$:

- a certain moduli space of meromorphic Higgs bundles on X
- a family of generalised Prym varieties for a family of (branched) W-Galois covers of X.

Both (have connected components which) are ACIHS in the Poisson sense. The first family is known as the "generalised" or "ramified" Hitchin system (with singular fibres removed). The second family is the "abelianisation" of the first one, and is (locally on the base) isomorphic it. While globally different, they have the same infinitesimal period map, and we shall use the second family for our main Kodaira–Spencer calculation.

We remark that the Hitchin base $\mathcal{B}$ depends on G, but only via $\mathfrak{g}$, and we write $\mathcal{B}_{\mathfrak{g}}$ whenever it is important to emphasise this dependence. There are certain loci in $\mathcal{B}$ for which we introduce special notation: the Zariski-open locus $\mathscr{B} \subset \mathcal{B}$ of generic cameral covers, $\mathcal{B}_0 \subset \mathcal{B}$ for the locus (25) of pluri-differentials vanishing along D, and $\mathbf{B} \subset \mathscr{B}$ for the base (26) of the integrable system, obtained by restricting the Hitchin map to a maximal rank symplectic leaf.

2. Meromorphic G-Higgs bundles

In this section we introduce our main objects of study: G-Higgs bundles on X with values in a vector bundle. Next, we discuss the main global properties of the coarse moduli space of $K_X(D)$-valued G-Higgs bundles. Finally, we study in more detail the locus in the moduli space, corresponding to Higgs bundles whose underlying principal bundle is regularly stable.

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