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Projective limits of state spaces I. Classical formalism[☆]

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ABSTRACT

In this series of papers, we investigate the projective framework initiated by Jerzy Kijowski (1977) and Andrzej Okołów (2009, 2013, 2014), which describes the states of a quantum (field) theory as projective families of density matrices. A short reading guide to the series can be found in [27].

The present first paper aims at clarifying the classical structures that underlies this formalism, namely projective limits of symplectic manifolds [27, subsection 2.1]. In particular, this allows us to discuss accurately the issues hindering an easy implementation of the dynamics in this context, and to formulate a strategy for overcoming them [27, subsection 4.1].

1. Introduction

An important step toward the quantization of a classical theory is the choice of a home for the kinematical quantum states: typically, we look for an Hilbert space supporting a representation of an algebra of selected kinematical observables. As long as we only deal with finitely many degrees of freedom, a comprehensive survey of the available options might still be within reach. But the extent and implications of this initial choice tend to get dramatically more involved in the case of field theory, where the huge algebra of kinematical observables can give rise to an intricate forest of representations. Unfortunately, it is hard to concisely formalize which requirements the elected representation should satisfy. In the worst case, we are left with the 'trial and error' method: pick some representation with attractive properties and check whether the next steps of the quantization program work well on it.

These next steps can fail for various reasons, one of them being that we committed ourselves to a space of kinematical states that, at a closer look, does not support the states we are really interested in. In particular, the space of physical quantum states, solving the dynamical constraints of the theory, should be rich enough. Refined Algebraic Quantization [1,2] is a way

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to look for physical states out of the representation we initially choose as the space of kinematical states; however, there are unfavorable situations, where we do not really know how to construct the additional input it requires.

Also, our space of states should contain the 'coherent states' needed to explore the semi-classical limit of the theory: we would like to associate to any point in the classical phase space a corresponding quantum state, suitably peaked around that point (see [3–6] for a discussion of this problem in the case of Loop Quantum Gravity [29], together with possible ways to circumvent it).

These issues motivate the search for alternative ways of building the space of kinematical states. Here, we will focus on a formalism first introduced by Jerzy Kijowski in the late 70s [7] and further developed by Andrzej Okołów recently [8–10]. The idea is to work in a setting that is more general than Hilbert spaces, and allows us to rely more heavily on the physical interpretation of the kinematical observables, namely how they are measured in practice. This tends to give state spaces that are bigger, but nevertheless technically easier to handle. In particular, we thus start with better chances to find the particular states we are looking for.

In the present work, we try to develop this formalism at a fairly general level [27], going beyond the extensive studies that have been carried out in special cases so far. To this intent, we will start by a detailed exposition of projective limits of symplectic manifolds, that build the natural classical counterpart of the quantum state spaces we want to discuss. An important observation is that such projective limits admit, at least locally, a preferred factorized description (Proposition 2.10). Therefore, we will look more closely at those projective systems where the factorization holds globally: not only they are often more convenient, they also reflect the core properties of the structures we are considering, so they are well-suited to get a first hold of complex questions. This will be in particular comfortable when turning to the quantum formalism in [11], but we will always try to sketch some ideas on how to strengthen those of our results that make explicit use of such a global factorization.

Specific difficulties arise when trying to implement the dynamics in this approach to (quantum) field theory. In Section 3, we will take advantage of having at our disposal a classical precursor of the formalism to analyze this question without having to deal at the same time with the inherent subtleties of the quantum dynamics. We will outline a suitable strategy, with the aim of doing justice both to the deep physical meaning of the issues at hand and to their practical significance for computations. This strategy will be applied to two simple toy models in [12].

Unless otherwise stated, all manifolds will be smooth manifolds, symplectic structures on them will be smooth, all maps between them will be smooth, and all submanifolds will be regular (i.e. embedded) submanifolds [15,28,30]. Where infinite dimensional manifolds are considered, these are Banach-modeled smooth manifolds, and symplectic structures on them are always strong symplectic structures [13, chap. VII].

2. Projective limits of classical phase spaces

The aim of this section is to describe the classical structures that, while underlying the constructions considered in previous works [7–10], have not been explicitly analyzed so far. The discussion of the physical interpretation will follow closely the one that has been given in these references.

The idea of the projective framework is to assemble a complicated classical theory (typically a field theory) from a collection of easier, smaller, truncated classical theories, by appropriately sewing them together. The motivation for this is twofold.

From the physical point of view, even when considering a theory with an infinite number of degrees of freedom, any given realistic experiment will involve only a finite number of observables, since measuring an infinite number of observables would require infinite time as well as infinite memory space (in fact, this means that any experiment can only measure a finite number of *boolean* observables, but we will not be that radical here, and will be satisfied with small truncated theories that are described by finite dimensional phase spaces). We will therefore think of the small partial theories as spanned by a finite number of elementary degrees of freedom. By "elementary", we mean those that can be measured in one experimental step, hence the justification for the choice of a collection of truncations should ultimately come from a careful analysis of what concrete experiments actually measure.

From a technical point of view, the smaller and easier theories are meant to be a convenient arena to develop systematic ways of calculating physical predictions. Indeed, a theoretical model will then be optimally useful if it comes with finite algorithms prescribing how to compute, at a given precision, the outcome of any arbitrary experiment.

Note however that the intuitive understanding just sketched has some weak points. One of them is that, even if we are considering only finitely many observables, it might occur that the Poisson-algebra they are generating cannot live on a finite dimensional symplectic manifold. Another (related) issue concerns the formulation of deterministic predictions while considering only finitely many degrees of freedom out of a field theory. Our viewpoint here is that these problems should not be relevant for the *kinematical* observables (these are supposed to build an easy algebra, and the question of writing down predictions does not belong to the kinematical level). Therefore, we postpone this discussion to Section 3, where we will refine the present framework to take into account the dynamics.

2.1. Projective systems of classical phase spaces

Having a collection of partial theories is not enough, we need to say how to connect them together in a consistent way (i.e. we do not want our physical predictions to depend on the particular partial theory in which we computed them). To