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Spin groups of super metrics and a theorem of Rogers

Ronald Fulp

Department of Mathematics, North Carolina State University, Raleigh NC 27695, United States

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ABSTRACT

We derive the canonical forms of super Riemannian metrics and the local isometry groups of such metrics. For certain super metrics we also compute the simply connected covering groups of the local isometry groups and interpret these as local spin groups of the super metric. Super metrics define reductions $\mathcal{OS}_g$ of the relevant frame bundle. When principal bundles $\widetilde{\mathcal{S}}_g$ exist with structure group the simply connected covering group $\tilde{\mathcal{G}}$ of the structure group of $\mathcal{OS}_g$, representations of $\tilde{\mathcal{G}}$ define vector bundles associated to $\widetilde{\mathcal{S}}_g$ whose sections are "spinor fields" associated with the super metric g. Using a generalization of a Theorem of Rogers, which is itself one of the main results of this paper, we show that for super metrics we call body reducible, each such simply connected covering group $\tilde{\mathcal{G}}$ is a super Lie group with a conventional super Lie algebra as its corresponding super Lie algebra.

Some of our results were known to DeWitt (1984) using formal Grassmann series and others were known by Rogers using finitely many Grassmann generators and passing to a direct limit. We work exclusively in the category of G^∞ supermanifolds with G^∞ mappings. Our supernumbers are infinite series of products of Grassmann generators subject to convergence in the ℓ_1 norm introduced by Rogers (1980, 2007).

1. Introduction

The canonical form of a super Riemannian metric g has been known [1] for some time in the case that the supermanifold on which g is defined is modeled on supervector spaces of the form $\mathbb{R}^{m|n}$ where the components of the vectors in $\mathbb{R}^{m|n}$ are supernumbers defined as formal power series of products of Grassmann generators. In this case there are no convergence issues. There seems to be no corresponding result when the underlying space of supernumbers is a series of products of Grassmann generators defined by requiring that the series converge in the ℓ_1 norm defined by Rogers [2,3]. It is our intent to determine such canonical forms in this case and to compute the corresponding local isometry groups $\mathcal{G}$ along with their simply connected covering groups $\tilde{\mathcal{G}}$. To do this we require a generalization of a Theorem of Rogers. It turns out that using this generalization we can show that for a large class of super metrics, called body reducible super metrics, the simply connected covering group of each local isometry group is a super Lie group whose super Lie algebra is a conventional super Lie algebra. Moreover, each such covering group is a semi-direct product of an ordinary simply connected finite-dimensional Lie group with a group N which is an infinite-dimensional generalization of a nilpotent group called a quasi-nilpotent group by Wojtyn'ski' [4] who discovered groups of this type. When representations of factors of the semi-direct product $\tilde{\mathcal{G}}$ are known each such representation gives rise to a representation of the semi-direct product itself. In every case representations of $\tilde{\mathcal{G}}$ define induced vector bundles and we think of the local sections of such vector bundles as local spinor fields associated with the super metric. In a real sense these fields have a right to be called super spinor fields but the notion of super spin also arises in quantum field theory [5] where super spin is a quantum number obtained from supersymmetric extensions of

E-mail address: fulp@math.ncsu.edu.

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the Poincare algebra. Quantum numbers of this type have already proven to be useful in that context. We do not know of applications of the type of super spinor field discussed in our work here but it would seem that there should be applications to supergravity.

In this paper we also prove a generalization of a Theorem of Rogers. We believe this result has significance in its own right quite aside from its present application to the program outlined above. We show that if $\mathfrak{g} = \mathfrak{g}^0 \oplus \mathfrak{g}^1$ is a $\mathbb{Z}_2$ graded Banach Lie algebra and $\mathfrak{h} = (\Lambda \otimes \mathfrak{g})^0$ is the even part of corresponding conventional super Lie algebra $\Lambda \otimes \mathfrak{g}$ then the kernel of the body mapping from $\mathfrak{h}$ into $\mathfrak{g}^0$ is a quasi-nilpotent Banach Lie algebra in the sense of Wojtyn'ski [4]. This Lie algebra has a global Banach Lie group structure defined via the exponential mapping and so is a quasi-nilpotent Banach Lie group. If we denote this group by N and if $\tilde{G}$ is the unique simply connected finite-dimensional Banach Lie group having $\mathfrak{g}^0$ as its Lie algebra, then we prove that there is an action of $\tilde{G}$ on N which defines a semi-direct product structure $H = \widetilde{G \times} N$. Moreover there is a G^∞ atlas on H relative to which H is a G^∞ super Lie group and this super Lie group has $\mathfrak{h}$ as its corresponding super Lie algebra $\mathfrak{h}$.

Rogers [3] proves a version of this Theorem in her book. She shows that in the case that the supernumbers are generated by a finite number of Grassmann generators then this result holds. In that case one has at ones disposal results from the theory of finite-dimensional Lie groups and Lie algebras. She is able to recover an infinite-dimensional version of her theorem by passing to a direct limit. She therefore has an analytic version of her theorem which utilizes the direct limit topology.

In our version of the Theorem we stick with G^∞ structures rather than analytic structures and we use Rogers ℓ_1 topology at various stages of our proof. Our proof follows hers but differs in details since it necessarily requires the appropriate modifications in order to remain within the category of infinite-dimensional Banach Lie groups and Banach Lie algebras.

In Section 2 we lay out our preliminary definitions and conventions. We follow the conventions of Rogers [3] in the main, but at times use slightly different notation. In subsections 1 and 2 of Section 3 we define the different norms required to make contact with the results of Wojtyn'ski's [4] . In subsection 3 of Section 3 we prove our version of Rogers theorem. In the fourth and final section of the paper we present our results on super Riemannian metrics.

2. Preliminaries

In this section we briefly introduce our conventions and assumptions regarding supernumbers, superspace, supermanifolds and basic constructions.

The starting point is a Grassmann algebra with coefficients in the field $\mathbb{K}$ which denotes either the field of real numbers or the field of complex numbers. We assume the Grassmann algebra is countably infinitely generated with generators $\zeta^1, \zeta^2, \zeta^3 \cdots$ which are anticommuting indeterminates;

$$\zeta^i \zeta^j = -\zeta^j \zeta^i. \tag{1}$$

As usual if $i = j$ the square of any Grassmann generator is zero.

Although we sometimes are not explicit regarding our convention we utilize a multi-index notation $I = (i_1, i_2, \ldots, i_k)$ with $1 \le i_1 < i_2 < \cdots < i_k$ and denote the set of all increasing strings of positive integers of length k by $\mathcal{I}_k$. We write

$$z = \sum_{p=0}^{\infty} \sum_{I \in \mathcal{I}_p} z_I \zeta^I \tag{2}$$

where for $I \in \mathcal{I}_k$,

$$\zeta^I = \zeta^{(i_1, i_2, \ldots, i_k)} = \zeta^{i_1} \zeta^{i_2} \ldots \zeta^{i_k} \tag{3}$$

$$z_I = z_{i_1, i_2, \ldots i_k} \in \mathbb{K}, \quad I \in \mathcal{I}_k, \quad k \ge 1 \tag{4}$$

and

$$z_I = z_0 \in \mathbb{K}, \quad I \in \mathcal{I}_0. \tag{5}$$

Notice that, for convenience, wherever the null-index $I \in \mathcal{I}_0$ appears as an index on an element it is denoted more simply by the label "0".

Now the set Λ of all supernumbers is, by definition, the set of all formal power series

$$z = \sum_{p=0}^{\infty} \sum_{I \in \mathcal{I}_p} z_I \zeta^I \tag{6}$$

such that

$$\|z\| = \sum_{p=0}^{\infty} \sum_{I \in \mathcal{I}_p} |z_I|$$