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# A numerical optimization approach to generate smoothing spherical splines

L. Machado [a,*], M. Teresa T. Monteiro [b]

[a] Department of Mathematics, University of Trás-os-Montes e Alto Douro (UTAD), 5001-801 Vila Real, Portugal
[b] ALGORITMI Research Centre, Department of Production and Systems, University of Minho, 4710-057 Braga, Portugal

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ABSTRACT

Approximating data in curved spaces is a common procedure that is extremely required by modern applications arising, for instance, in aerospace and robotics industries.

Here, we are particularly interested in finding smoothing cubic splines that best fit given data in the Euclidean sphere. To achieve this aim, a least squares optimization problem based on the minimization of a certain cost functional is formulated. To solve the problem a numerical algorithm is implemented using several routines from MATLAB toolboxes. The proposed algorithm is shown to be easy to implement, very accurate and precise for spherical data chosen randomly.



## 1. Introduction

This paper is essentially addressed to the numerical computation of smoothing cubic splines in the spherical surface. Contrary to what happens in the Euclidean spaces, where closed formulas for smoothing splines can be easily obtained using algebraic operations [1], such is not the case for general curved spaces. Problems of fitting curves and surfaces date back from several millennia ago and appeared firstly in astronomy to determine the shape and size of celestial bodies and their trajectories [2]. In the eighteenth century, the contribution of Gauss with the theory of least squares was a remarkable landmark to solve those problems [3]. In linear least squares problems, it is given a distributed set of nodes in the plane and the goal is to find a global model, typically a polynomial of a certain degree, that best fits the data. However, this global model is less adaptive to local variations. This might be more realistic when the data comes from experimental tasks. It might happen that some individual observations can have undue influence in the fitted curve and this can lead to drastic variations on it. This limitation can easily be overcome by using piecewise polynomials, the basic idea behind smoothing splines. It turns out that these curve fitting techniques are more versatile and powerful than the global least squares models [4].

* Corresponding author.
*E-mail address:* lmiguel@utad.pt (L. Machado).

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Generalizing fitting problems to more general curved spaces received more attention in the late eighties due to the development of mechanical and robotics industries. Since the pioneering work on spherical smoothing splines [5], estimation of functions defined on non-Euclidean spaces is widely recognized as an important problem in diverse areas of research such as image processing [4,6], trajectory planning problems [7], geophysics [5] and in data analysis in general [8].

The most difficulty when generalizing the classical fitting problems to more general curved spaces is the lack of knowledge of the expressions for polynomial curves in such spaces. The most common definition of a geometric polynomial, that we also adopt in this work, is based on a variational approach [9–11]. Although there is a wide range of literature covering the generalization of interpolating or smoothing splines to curved spaces [5,9,10,12–15], too little has been done from a numerical point of view. We refer the two recent works [14,15], where a numerical algorithm based on a gradient steepest descent method has been proposed to generate smoothing splines in some curved spaces. In these works, the authors used a Palais-metric in the infinite dimensional space of all the admissible curves and then apply a steepest descent algorithm to generate the splines. The algorithm that comes up in this work to generate spherical cubic splines follows a totally different perspective since it takes only into account the intrinsic geometry of the spherical surface. Moreover, it is easy to implement and it is proved to be accurate, robust and efficient. We believe that this algorithm can be an asset within the scientific community to solve occasional problems that require the practical computation of spherical smoothing splines.

The paper is organized as follows. In Section 3, we recall the classical least squares problem and present its generalization to more general curved spaces. To carry out this generalization, several notions from differential geometry are introduced. In Section 4, we consider a constrained nonlinear optimization problem in the embedding Euclidean space whose solutions are smoothing cubic splines in the Euclidean sphere. The necessary optimality conditions for this optimization problem are derived in Theorem 4.1. A numerical algorithm to find approximate solutions for the nonlinear optimization problem is therefore proposed in Section 4.3 and some of the numerical results are presented in Section 5. Finally, some concluding remarks about this work and ideas for future work are carried out in Section 6.

## 2. Motivation and applications

The astounding development of the mechanical and robotics industries required the generalization of classical techniques to more general curved spaces. Among non-Euclidean spaces, the two dimensional sphere $S^2$ has been one of the major attractions driven by obvious applications in the earth sciences and meteorology.

In particular, spherical splines have potential applications not only in the earth sciences but also in medicine, biology, astronomy, computer graphics, animation, robotics and in motion planning based on quaternions. Spherical data occurs in vector cardiograms, where the information about the electrical heart activity during a heartbeat is described in terms of a near-planar orbit in $\mathbb{R}^3$. Another example is, for instance, the landmark-based shape analysis of objects, where 2D objects are represented by configurations of landmarks, being the unit sphere $S^{2k-1}$ the set of all such configurations (for configurations with $k$ landmarks) [16].

Since the very first naive description of spherical splines based on a normalization procedure introduced by Parker and Denham [17], there has been an extensive work proposing a wide variety of techniques to generate spherical splines and establishing their theoretical properties.

Thompson and Clark [18] applied for the first time spherical splines to the problem of determining the Gondwanan apparent polar wonder path to study the past movements of continents.

Spherical splines on the unit sphere $S^3$ have become popular in computer graphics and animation because of the correspondence between orientations of solid bodies and quaternions. Quaternions can be viewed as pairs of antipodal points on $S^3$ and thus spherical spline curves can be used to specify a smooth transition of solid orientations [19].

For an interesting survey on spherical splines we refer to [20], where it is pointed out several descriptions of spherical splines that appeared along the history together with the main advantages and disadvantages in choosing one or another definition.

With the arrival of the 3D photography, function estimation of spherical data received a renewed impetus [21]. The goal of the 3D photography is to generate computer models of physical objects reflecting their shape and other aspects that include color and reflectance. Such features can be naturally seen as functions defined on a surface that have to be estimated from physical measurements.

Spherical splines find themselves applicable in brain surface representation and analysis. The outward folds and the cortical grooves of the human cortical surface encode important anatomical features which provide a splitting of the cortex surface into anatomically distinct areas. Surface-based modeling is valuable in brain imaging not only to help analyze anatomical shape but also to statistically combine or compare 3D anatomical models across subjects [6].

The examples listed above are only a few among the numerous areas where spherical splines have applicability. The main concern of this paper is to present a simple and efficient numerical algorithm to generate smoothing spherical splines in order to be used within the scientific community that works with practical applications of spherical splines.

## 3. Mathematical framework

### 3.1. Linear least squares problem

In Euclidean spaces, the problem of finding a polynomial curve that best fits given data (points and times) is well known [22] and it can be defined by the following. Let $p_0, \ldots, p_k$ be a set of $k+1$ distinct points in $\mathbb{R}^n$ and let us associate to each