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The classification of the Weyl conformal tensor in 4-dimensional manifolds of neutral signature

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ABSTRACT

This paper presents a simple account of the algebraic classification of the Weyl conformal tensor on a 4-dimensional manifold with metric g of neutral signature $(+, +, -, -)$. The classification is algebraically similar to the well-known Petrov classification in the Lorentz case and the various algebraic types and corresponding canonical forms are obtained. Criteria concerning principal, totally null 2-spaces are explored and which lead to principal null directions similar to those of L. Bel in the Lorentz case. The uniqueness, or otherwise, of the tetrads in which the canonical forms appear are investigated and some topological and differentiability properties of the algebraic types are also established.

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1. Introduction

The aim of this paper is to present a classification of the Weyl conformal tensor C on a 4-dimensional smooth manifold admitting a smooth metric g of neutral signature $(+, +, -, -)$. After the work was completed the author's attention was drawn to the papers of Law [1,2], Batista [3] and Ortaggio [4] in which some of the algebraic results presented here at the beginning of Section 3 are given (and mostly in spinor language in [1,2]). Another approach to this problem has been discussed in [5]. However, it is believed that the methods adopted in this paper, which use a standard tensor approach, are simpler, sharper structured and more amenable and convenient for differential geometers and for purposes of calculation. They lead to a full classification of C. Further, the idea of a principal, totally null 2-space is introduced for the self dual and anti-self dual parts of C and shown to lead directly to the concepts of repeated and non-repeated principal null directions for the full tensor C, the equivalents of which have been rather useful in the Lorentz case for general relativity theory. In addition, the (possible) lack of uniqueness (up to reflections, etc.) of the canonical tetrad in which each type for C is expressed is given in full. Some topological and differentiability properties of the classification are also derived.

To establish notation, M denotes a 4-dimensional, smooth manifold with smooth metric g of neutral signature $(+, +, -, -)$ and, collectively, these are labelled (M, g). The tangent space at $m \in M$ is denoted by T_mM and the vector space of 2-forms (usually referred to as *bivectors*) at m by Λ_mM. Due to the existence of the metric (and where no confusion could arise) the distinction between the tangent and cotangent spaces will sometimes be ignored as will the index placing on bivectors. The symbol $u.v$ denotes the inner product at m, $g(m)(u, v)$, of $u, v \in T_mM$. A non-zero member $u \in T_mM$ is called *spacelike* if $u.u > 0$, *timelike* if $u.u < 0$ and *null* if $u.u = 0$. The symbol $*$ denotes the usual Hodge duality (linear) operator (on Λ_mM) and square brackets round indices denote the usual anti-symmetrisation of the indices enclosed. Since g has

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neutral signature one may choose a pseudo-orthonormal basis x, y, s, t at $m \in M$ with $x.x = y.y = -s.s = -t.t = 1$ and an associated *null* basis of (null) vectors l, n, L, N at m given by $\sqrt{2}l = x + t$, $\sqrt{2}n = x - t$, $\sqrt{2}L = y + s$ and $\sqrt{2}N = y - s$ so that $l.n = L.N = 1$ and all other such inner products are zero. The associated completeness relations are $g_{ab} = x_a x_b + y_a y_b - s_a s_b - t_a t_b = l_a n_b + n_a l_b + L_a N_b + N_a L_b$. A 2-dimensional subspace (2-space) V of $T_m M$ is called *spacelike* if each non-zero member of V is spacelike, or each non-zero member of V is timelike, *timelike* if V contains exactly two, null 1-dimensional subspaces (*directions*), *null* if V contains exactly one null direction and *totally null* if each non-zero member of V is null. Thus a totally null 2-space consists, apart from the zero vector, of null vectors any two of which are orthogonal. This list is mutually exclusive and exhaustive. A bivector E at m with components $E^{ab}(= -E^{ba})$ necessarily has even matrix rank. If this rank is 2, E is called *simple* and if 4, it is called *non-simple*. If E is simple it may be written $E^{ab} = u^a v^b - v^a u^b$ for $u, v \in T_m M$ and the 2-space spanned by u and v is uniquely determined by E and called the *blade* of E (and then, unless more precision is required, E or its blade is written $u \wedge v$). A simple bivector is called *spacelike* (respectively, *timelike*, *null* or *totally null*) if its blade is *spacelike* (respectively, *timelike*, *null* or *totally null*). The set $\Lambda_m M$ admits a (bivector) metric denoted by P so that for $E, E' \in \Lambda_m M$, $P(E, E') \equiv P_{abcd} E^{ab} E'^{cd} = E^{ab} E'_{ab}$ where $P_{abcd} = \frac{1}{2}(g_{ac} g_{bd} - g_{ad} g_{bc})$. Sometimes one writes $|E|$ for $P(E, E)$. It is easily checked that, as a consequence of the neutral signature of g, the signature of P is $(+, +, -, -, -, -)$.

For each $E \in \Lambda_m M$ its dual is defined by $\overset{*}{E}_{ab} = \frac{1}{2}\epsilon_{abcd} E^{cd}$ where $\epsilon_{abcd} \equiv \sqrt{detg}\,\delta_{abcd}$ with δ denoting the usual alternating symbol and $detg$ the determinant of g. For neutral signature $\overset{**}{E} = E$ and so the only eigenvalues of the linear map $*$ are ± 1. Now define the subspaces $\overset{+}{S}_m \equiv \{E \in \Lambda_m M : \overset{*}{E} = E\}$ and $\overset{-}{S}_m \equiv \{E \in \Lambda_m M : \overset{*}{E} = -E\}$ and also the subset $\widetilde{S}_m \equiv \overset{+}{S}_m \cup \overset{-}{S}_m$, of $\Lambda_m M$. Then $\overset{+}{S}_m \cap \overset{-}{S}_m = \{0\}$ and $E \in \Lambda_m M \setminus \widetilde{S}_m$ if and only if E and $\overset{*}{E}$ are independent members of $\Lambda_m M$. If E and E' are independent and totally null and both are in $\overset{+}{S}_m$ or both are in $\overset{-}{S}_m$ their blades intersect in only the zero vector whereas if $E \in \overset{+}{S}_m$ and $E' \in \overset{-}{S}_m$ their blades intersect in a unique null direction. It follows that if $k \in T_m M$ is null there are exactly two totally null 2-spaces containing k, one in $\overset{+}{S}_m$ and one in $\overset{-}{S}_m$. Any $E \in \Lambda_m M$ may be written in exactly one way as $E = \overset{+}{E} + \overset{-}{E}$ with $\overset{+}{E} \in \overset{+}{S}_m$ and $\overset{-}{E} \in \overset{-}{S}_m$ and so one has $\Lambda_m M = \overset{+}{S}_m \oplus \overset{-}{S}_m$. It follows that if E is a *null* bivector its unique decomposition $E = \overset{+}{E} + \overset{-}{E}$ gives rise to unique totally null members $\overset{+}{E}$ and $\overset{-}{E}$ whose unique common null direction equals that of E and conversely the sum of any pair of totally null bivectors $\overset{+}{E} \in \overset{+}{S}_m$ and $\overset{-}{E} \in \overset{-}{S}_m$ gives a null bivector whose unique null direction equals the intersection of the blades of $\overset{+}{E}$ and $\overset{-}{E}$. Also if $\overset{+}{E} \in \overset{+}{S}_m$ and $\overset{-}{E} \in \overset{-}{S}_m$, one has $P(\overset{+}{E}, \overset{-}{E}) = 0$ and $[\overset{+}{E}, \overset{-}{E}] = 0$ where [ ] denotes matrix commutation. Now each of $\overset{+}{S}_m$ and $\overset{-}{S}_m$ is Lie isomorphic to the Lie algebra $o(1, 2)$ under [ ] and so $\Lambda_m M$ is the Lie algebra product $\overset{+}{S}_m \oplus \overset{-}{S}_m$. Clearly $E \in \Lambda_m M$ is simple if and only if $\overset{*}{E}$ is. It is sometimes useful to note that for a general non-zero member $E \in \Lambda_m M$ the statements that (i) E is simple, (ii) $E_{ab}\overset{*}{E}{}^{bc} = 0$ and (iii) $E_{ab}\overset{*}{E}{}^{ab} = 0$ are equivalent. It can also be checked that for $E \in \widetilde{S}_m$, the statements that $|E| = 0$, that E is simple and that E is totally null are equivalent. A bivector E not in $\widetilde{S}_m$ is null if and only if it is simple with a null vector l in its blade satisfying $E_{ab} l^b = 0$. More details on some of these matters may be found in [6–10].

2. The classification of the Weyl tensor I

The Weyl conformal tensor C is a type $(1, 3)$ tensor with components $C^a{}_{bcd}$ introduced by Weyl [11] and has the property that any two conformally related metrics on M have the same Weyl conformal tensor independently of M and of the metric signature provided that $\dim M \geq 3$. Then C leads to a related type $(0, 4)$ tensor (also denoted by C) with components $C_{abcd} \equiv g_{ae} C^e{}_{bcd}$ and the skew-symmetry of this latter tensor in its first and last pairs of indices leads to the possibility of left and right duals for C, one for each such pair. The fact that C also satisfies the trace-free condition $C^c{}_{acb} \equiv 0$ means that the convenient relationship ${}^*C^* = C$ or, equivalently, ${}^*C = C^*$ is satisfied (see, e.g. [12]). The tensor C also satisfies $C_{a[bcd]} = 0$ which is equivalent to ${}^*C$ being tracefree, ${}^*C^c{}_{acb} = 0$, and since C is tracefree, ${}^*C_{a[bcd]} = 0$. Now consider the linear map f on $\Lambda_m M$ (here called the *Weyl map*) given by $E^{ab} \to C^{ab}{}_{cd} E^{cd}$. The above dual relations show that the subspaces $\overset{+}{S}_m$ and $\overset{-}{S}_m$ are invariant subspaces for f. Now write the tensor type $(0, 4)$ relation

$$C = \overset{+}{W} + \overset{-}{W} \qquad \overset{+}{W} \equiv \frac{1}{2}(C + {}^*C), \qquad \overset{-}{W} \equiv \frac{1}{2}(C - {}^*C). \tag{1}$$

Thus the respective self dual and anti-self dual parts of C, $\overset{+}{W}$ and $\overset{-}{W}$, satisfy $\overset{+}{W}{}^* = \overset{+}{W}$ and $\overset{-}{W}{}^* = -\overset{-}{W}$ and give rise in an obvious way to maps $\overset{+}{f}$ and $\overset{-}{f}$ constructed from them, as f was from C, such that $f = \overset{+}{f} + \overset{-}{f}$. Thus $\overset{+}{f} : \overset{+}{S}_m \to \overset{+}{S}_m$ and $\overset{+}{f} : \overset{-}{S}_m \to \{0\}$ and $\overset{-}{f} : \overset{-}{S}_m \to \overset{-}{S}_m$ and $\overset{-}{f} : \overset{+}{S}_m \to \{0\}$. The tensors $\overset{+}{W}$ and $\overset{-}{W}$ are tracefree, satisfy $\overset{+}{W}_{a[bcd]} = \overset{-}{W}_{a[bcd]} = 0$ and are uniquely determined by C.

Following a duality convention on the basis (l, n, L, N) for $T_m M$ one may select as a basis for $\overset{+}{S}_m$ the set $F \equiv l \wedge n - L \wedge N$, $G \equiv l \wedge N$ and $H \equiv n \wedge L$ satisfying the orthogonality relations $|F| = -4$, $|G| = |H| = P(F, G) = P(F, H) = 0$, $P(G, H) = 2$.

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