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On the realizable topology of a manifold with attractors of geodesics

Marcio Colombo Fenille

Faculdade de Matemática - Universidade Federal de Uberlândia, Av. João Naves de Ávila 2121, Santa Mônica, 38408-100, Uberlândia MG, Brazil

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ABSTRACT

We discuss the so-called realizable topology of a Riemannian manifold with attractors of geodesics, which we understand as its topological properties, mainly that related to its fundamental group, investigated from a viewpoint that may be considered realizable in a sense. In the special approach in which the manifold is understood as a model physical universe, we conclude that its realizable fundamental group is isomorphic to the classical fundamental group of its observable portion. For a universe of dimension at least three whose unobservable components are all contractible, this conclusion ensures the possibility to get real inferences about its classical fundamental group through observational methods.

1. Introduction

One of the keys to understand the topology of the universe is to know its fundamental group. Despite the massive research on the subject, Science has no answer, for instance, for the question: Is the universe simply connected or multiply connected? From the viewpoint of the Cosmology, this question has been approached via observational methods; see [1]. In this sense, many topological models for the universe have been proposed and tested. For instance, in [2] it is presented a study of a hyperbolic universe with horned topology, in [3] it is tested the so-called Poincaré dodecahedral space topology hypothesis, and the research continues; see [4]. However, some underlying questions may be being neglected: It is possible to get the topology of the whole universe through observational methods? The observable portion of the universe contains sufficient information on the topology of the whole universe?

Problems like that show the importance of understanding the range of a scientific method and the types of problems that a theory is able to treat. Now, the investigation about the range of a scientific theory or method must be based on mathematical foundation, specially mathematical theories which are able to answer questions about existence and feature. This article presents such an investigation, with a strictly mathematical approach, via Algebraic, General and Differential

E-mail address: mcfenille@gmail.com.

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Topology, which leads us to important conclusion about the specific question: Can the fundamental group of the whole universe be detected by considering only information about the observable portion of the universe?

In order to reconcile the language, we define what we call the realizable fundamental group of a metric space, and so we establish comparison between that and the classical fundamental group. To explain the idea, let suppose that we want to study some topological property of a space X that may be formulated in terms of paths or homotopy of paths, as path connectedness and fundamental group, but for some reason we are only interested in a specific set of paths. Thus, given two paths in such a set, we would like to investigate, for example, the existence of a homotopy between them, but also a specific homotopy, namely, a homotopy that in each stage is also a path belonging to the specified set. Such an investigation gives information on the (algebraic) topology of the space, but from the nature of such information, we will say that they refer to the *realizable topology* of X according to the specification made *a priori*. For instance, let us say that we want to study the fundamental group of a pointed space (X, x_*), but not considering all possible loops in this space, but just those of a special subset $\mathcal{G}$ of the set of the all loops in (X, x_*). It may happen that, in this way, we would never know completely the fundamental group of (X, x_*), unless the set $\mathcal{G}$ is sufficiently representative of the set of all loops in (X, x_*). Then, it does not make sense to denote by $\pi_1(X, x_*)$ the object that results from the proposed analysis. Therefore, we denote it by $\pi_1^{\mathcal{G}}(X, x_*)$ and we say that it is the *realizable fundamental group of X according to $\mathcal{G}$*.

In the case in which X is a model of the physical universe and $\mathcal{G}$ is the set of the paths that a photon may realize (which we call a light path), it is possible that the homotopy classes of such paths do not determine the fundamental group of X. But if the only phenomena which we may observe are the light paths, then these classes determines the so-called *observable fundamental group* of the universe.

From next section, we develop a study guided by these ideas, with a purely mathematical approach, but always in order to seeking to draw comparisons with a model of the physical universe and to answer to underlying questions presented above. The finding concerning this are presented in the final section of the article.

Because of our proposal and goal, we consider, from the beginning, the space of study as being a connected and locally path connected metric space X, which latter is replaced by a connected Riemannian manifold.

In Section 2 we introduce the basic necessary concepts, namely, the concepts of prime paths, admissible paths and attractors of prime primes. In our physical model, these concepts play the role of light paths, observable paths and unobservable components of the universe, respectively.

In Section 3 we study the connectedness of a space according to prime/admissible paths. We establish a complete relationship between this type of connectedness and both the "size" of the set of admissible paths and the existence of attractors of prime paths. In our physical model, these relationships are decisive to establish the attractors of prime paths as the unobservable components of the universe.

In Section 4 we define, for a given space and a given set $\mathcal{G}$ of prime paths, the so-called realizable fundamental group of the space according to the set $\mathcal{G}$.

In Section 5 we consider a Riemannian manifold X and the set $\mathcal{G}^*$ of all oriented geodesics in X. We show that $\mathcal{G}^*$ is a set of prime paths in X respect to which X is $\mathcal{G}^*$-connected and has no $\mathcal{G}^*$-attractor, which implies that such a model is not good to draw comparisons with the physical universe. On the other hand, we prove that if we requires that X has a well-behaved collection of attractors given *a priori*, then we may build a subset $\mathcal{G}^*_{\mathbf{m}} \subset \mathcal{G}^*$ of prime paths fulfilling our requirement.

In Section 6 we study the realizable fundamental group of a Riemannian manifold with attractors of geodesics, according to the set of prime paths $\mathcal{G}^*_{\mathbf{m}}$ built in the main theorem of Section 5. In Section 7 we consider the special case in which all the $\mathcal{G}^*_{\mathbf{m}}$-attractors are contractible; we relate the realizable fundamental group and the classical fundamental group by way of a exact sequence of fundamental groups.

Finally, in Section 8 we use the theoretical concepts and results presented in Sections 1–7 to draw conclusions about the so-called observable fundamental group of the universe, which we understand as the realizable fundamental group of the universe according to observable paths. We finding that the observable fundamental group of the universe coincides with the classical fundamental group of its observable portion, but it does not coincide necessarily with the classical fundamental group of the whole universe. However, if the universe has dimension at least three and all the components of its unobservable portion are contractible, then the observable fundamental group of the whole universe is isomorphic to its classical fundamental group. Therefore, in this case, it is possible that we can get information about the classical fundamental group of the whole universe via observational methods.

2. Attractors of prime paths in a metric space

Throughout this section, X is a connected and locally path connected (and so path connected) metric space.

If $\sigma : [a, b] \to X$ is a path in X from x to y, the path $\sigma^{-1} : [a, b] \to X$ defined by $\sigma^{-1}(t) = \sigma(a + b - t)$ is a path in X from y to x, called the reverse path of σ or the path obtained from σ by reversing its orientation.

Although always exists in X a path joining any two points, it may happen that for some particular pair of points there is not a path of a certain specific class joining them. For instance, if X is a Riemannian manifold corresponding to a model of the physical universe, although always exists mathematically in X paths connecting any two points, it may occur that such a path is not physically realizable in some sense. Moreover, it may occur that exists a physically realizable path from x to y, but there is not a physically realizable path from y to x. We discuss this problem in a more clear formulation. For this, we start by considering the set ΩX of all (continuous) paths in X, defined in some (possibly degenerated) compact interval of the real line. All the paths given *a priori* will be considered belonging to ΩX.