Physica D 359 (2017) 1–20

Contents lists available at ScienceDirect

Physica D

journal homepage: www.elsevier.com/locate/physd

Dynamical systems analysis of the Maasch–Saltzman model for glacial cycles

Hans Engler [a,b], Hans G. Kaper [a,b], Tasso J. Kaper [c,*], Theodore Vo [c]

[a] Department of Mathematics and Statistics, Georgetown University, Washington, DC 20057, United States
[b] Mathematics and Climate Research Network (MCRN)[1]
[c] Department of Mathematics and Statistics, Boston University, Boston, MA 02215, United States

HIGHLIGHTS

- 2D slow- and center manifolds govern dynamics in Maasch–Saltzman model.
- Bogdanov–Takens singularities organize local and global bifurcations.
- Regions of all stable limit cycles identified.
- Slow passage through Hopf causes delayed stability loss of key equilibrium.
- Delayed stability loss responsible for mid-Pleistocene transition mechanism.

ARTICLE INFO

Article history:
Received 30 March 2017
Received in revised form 4 August 2017
Accepted 14 August 2017
Available online 24 August 2017
Communicated by B. Sandstede

Keywords:
Glacial cycles
Maasch–Saltzman model
Center manifolds
Bogdanov–Takens singularities
Symmetry breaking
Slow passage through Hopf bifurcation

ABSTRACT

This article is concerned with the internal dynamics of a conceptual model proposed by Maasch and Saltzman (1990) to explain central features of the glacial cycles observed in the climate record of the Pleistocene Epoch. It is shown that, in most parameter regimes, the long-term system dynamics occur on certain intrinsic two-dimensional invariant manifolds in the three-dimensional state space. These invariant manifolds are slow manifolds when the characteristic time scales for the total global ice mass and the volume of the North Atlantic Deep Water are well separated, and they are center manifolds when these characteristic time scales are comparable. In both cases, the reduced dynamics on these manifolds are governed by Bogdanov–Takens singularities, and the bifurcation curves associated to these singularities organize the parameter regions in which the model exhibits glacial cycles. In addition, knowledge of the reduced systems and their bifurcations is useful for understanding the effects of slowly varying parameters, which cause passage through Hopf bifurcations, and of orbital (Milankovitch) forcing. Both are central to the mechanism proposed by Maasch and Saltzman for the mid-Pleistocene transition in their model.

1. Introduction

The dynamics of glacial cycles during the Pleistocene Epoch – the period from approximately 2.6 million years before present (2.6 Myr BP) until approximately 11.7 thousand years before present (11.7 Kyr BP) – are of great current interest in the geosciences community, see [1], [2, §11] and [3, §12.3]. The geological record shows cycles of advancing and retreating continental glaciers, mostly at high latitudes and high altitudes, and especially in the Northern Hemisphere. The typical temperature pattern inferred from proxy data resembles that of a sawtooth wave, where a slow glaciation is followed by a rapid deglaciation. In the early Pleistocene (until approximately 1.2 Myr BP), the period of a glacial cycle averaged 40 Kyr; after a transition period of approximately 400 Kyr, the glacial cycles had a noticeably greater amplitude and their period averaged 100 Kyr. Although the periods appear to correlate to the cycles of the orbital forcing (Milankovitch theory [4]), the evidence is subject to debate [5, § 11.8], and there is currently no widely-accepted explanation for the mid-Pleistocene transition, when the period of the cycles changed from approximately 40 Kyr to 100 Kyr. Several models have been proposed to explain the various observations; see, for example, [6–23]. We refer the reader to [24] for an overview of these various modeling efforts and to [1] for a general introduction to paleoclimate modeling. The present investigation focuses on the internal dynamics of the conceptual model developed by Maasch and Saltzman [13,20].

* Corresponding author.
E-mail address: tasso@math.bu.edu (T.J. Kaper).
1 https://mathclimate.org.

http://dx.doi.org/10.1016/j.physd.2017.08.006
0167-2789/

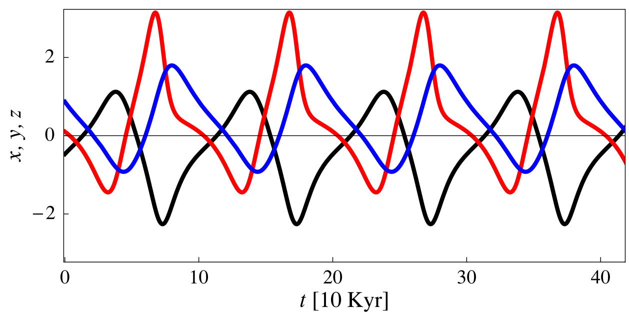

Fig. 1.1. Limit cycle of (1.1) at $p = 1.0$, $q = 1.2$, $r = 0.8$, $s = 0.8$. The three curves represent the total ice mass (black), atmospheric CO_2 concentration (red), and volume of NADW (blue).

1.1. The Maasch–Saltzman model

The Maasch and Saltzman (MS) model is based on physical arguments and emphasizes the role of atmospheric CO_2 in the development and evolution of the glacial cycles. In nondimensional form, it consists of the following three ordinary differential equations:

$$\begin{aligned} \dot{x} &= -x - y, \\ \dot{y} &= ry - pz + sz^2 - yz^2, \\ \dot{z} &= -qx - qz. \end{aligned} \tag{1.1}$$

The state variables x, y, and z represent the anomalies (deviations from long-term averages) of the total global ice mass, the atmospheric CO_2 concentration, and the volume of the North Atlantic Deep Water (NADW), respectively. The latter is a measure of the strength of the North Atlantic overturning circulation and thus of the strength of the oceanic CO_2 pump.

The nondimensional parameters p, q, r, and s are each combinations of various physical parameters, and they are all positive. Here, p and r represent the effective rate constants for how the CO_2 concentration (y) changes as the NADW (z) and CO_2 concentration change, respectively. Next, q is the effective ratio of the characteristic time scales for the total global ice mass (x) and the volume of NADW; for physical reasons, $q > 1$ [20]. Then, the parameter s is a symmetry parameter. With $s = 0$, the model (1.1) possesses a reflection symmetry; if (x, y, z) is a solution, then so is $(-x, -y, -z)$. In this special case, glaciation and deglaciation occur at the same rates. Physically, however, it is observed that deglaciation occurs at a faster rate than glaciation, and $s > 0$ guarantees this asymmetry. All of these nondimensional parameters incorporate several dimensional rate constants as well as dimensional parameters and quantities related to the global mean sea surface temperature and the mean volume of permanent sea ice. The full derivation of the model is given in [20, §2].

In [13], Maasch and Saltzman showed computationally that the model (1.1) exhibits oscillatory behavior with dominating periods of 40 Kyr in response to insolation forcing with such periods, and limit cycles with 100 Kyr periods if $p = 1$, $q = 1.2$, $r = 0.8$, and $s = 0.8$ in the absence of forcing. They also showed that a transition from a 40 Kyr cycle to a 100 Kyr cycle can be achieved by slowly varying the parameters p and r across a certain threshold.

Fig. 1.1 shows a representative 100 Kyr limit cycle. Each cycle is clearly asymmetric: a rapid deglaciation is followed by a slow glaciation. This asymmetry arises in (1.1) for $s > 0$. Also, the three variables are properly correlated: as the concentration of the atmospheric CO_2 (a greenhouse gas) increases, the climate gets warmer, and the total ice mass decreases (deglaciation); as the volume of NADW increases, the strength of the North Atlantic overturning circulation increases, more atmospheric CO_2 is absorbed by the ocean and, consequently, the atmospheric CO_2 concentration decreases.

1.2. Summary of the results

In this article, we present a dynamical systems analysis of the internal dynamics of the Maasch–Saltzman (MS) model (1.1). We study the model dynamics for all values $q > 1$ and $s \geq 0$; and, in each of the different regimes of the parameters q and s, we use the parameters p and r as the primary bifurcation parameters. In general terms, the main results are the identification of the Bogdanov–Takens (BT) points [25–29] in the (p, r) plane that act as organizing centers in the parameter space for all of the equilibria, limit cycles, homoclinic orbits, and their bifurcations.

The first set of results is for the *symmetric, slow–fast MS model*, which is obtained by setting $s = 0$ and taking $q \gg 1$ in (1.1). In this regime, the system is $(2 + 1)$-dimensional, with two slow variables x and y and one fast variable z. We show that there is a family of two-dimensional slow invariant manifolds along which the fast variable is slaved to the slow variables and to which all solutions quickly relax. We study the dynamics on the slow manifolds. The central feature is a $\mathbb{Z}_2$-symmetric BT bifurcation point, from which all bifurcation curves emanate. The curves of Hopf bifurcations, homoclinic bifurcations, and saddle–node bifurcations of limit cycles determine the regions in parameter space where the stable limit cycles exist. In addition, since all solutions relax quickly to the slow manifolds, one can determine the basins of attraction of the various limit cycles. These first results build naturally on the analysis of the symmetric MS model (1.1) in the limit $q = \infty$ [30].

The second set of results concerns the effects of asymmetry ($s > 0$). In the regime of asymptotically large values of q ($q \gg 1$), the system (1.1) with $s > 0$ is also a slow–fast system, with x and y as slow variables, and z as a fast variable. There is again a family of exponentially attracting two-dimensional, invariant slow manifolds, but the symmetry-breaking makes the dynamics on the slow manifolds more complex. With $s > 0$, the limit cycles observed in (1.1) are asymmetric, exhibiting a relatively rapid deglaciation and a relatively slow glaciation, as shown in Fig. 1.1. The lone $\mathbb{Z}_2$-symmetric BT point that exists for $s = 0$ splits into two generic BT points for $s > 0$, and we show how the Hopf bifurcations and homoclinic bifurcations emanating from these two BT singularities determine the boundaries of the domains of the stable and unstable limit cycles.

With these results in hand, we are then in a position to analyze and visualize the dynamics of the full, asymmetric ($s > 0$) MS model (1.1) for all finite values of $q > 1$, which is physically the most relevant regime. We show that for all $q > 1$ the system possesses a family of two-dimensional center manifolds toward which solutions relax. Moreover, the solutions of the full system may be accurately approximated by those of the reduced systems on the center manifolds for all $q > 1$, and the manifold is at least C^1-smooth for all q greater than a critical value $q_c(p, r, s)$. On the center manifolds, the system has a pair of BT singularities, just as in the asymmetric slow–fast MS model, and the bifurcation curves emanating from them organize the system dynamics, including the boundaries of the domains of existence of the stable and unstable limit cycles.

The final set of results concerns an initial investigation of the effects of slow, linear variation of the parameters p and r in the model (1.1), as well as of orbital forcing. We show that delayed passage through a Hopf bifurcation, which results from the slow parameter variation, drives the mechanism proposed by Maasch and Saltzman for the mid-Pleistocene transition, and that the forcing amplitude and frequency impact the duration of the delay.