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Controlling roughening processes in the stochastic Kuramoto–Sivashinsky equation

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HIGHLIGHTS

- Generic framework for the control of stochastic partial differential equations.
- Framework exemplified with the stochastic Kuramoto-Sivashinsky (sKS) equation.
- Control of roughening processes of sKS equation with Burgers or KPZ nonlinearity.
- Force solutions of stochastic partial differential equations to a prescribed shape.
- Controls are linear and can be both localized or periodically distributed.

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ABSTRACT

We present a novel control methodology to control the roughening processes of semilinear parabolic stochastic partial differential equations in one dimension, which we exemplify with the stochastic Kuramoto-Sivashinsky equation. The original equation is split into a linear stochastic and a nonlinear deterministic equation so that we can apply linear feedback control methods. Our control strategy is then based on two steps: first, stabilize the zero solution of the deterministic part and, second, control the roughness of the stochastic linear equation. We consider both periodic controls and point actuated ones, observing in all cases that the second moment of the solution evolves in time according to a power-law until it saturates at the desired controlled value.

1. Introduction

Roughening processes arise in nonequilibrium systems due to the presence of different mechanisms acting on multiple time and length scales and are typically characterized by a time-fluctuating "rough" interface whose dynamics are described in terms of a stochastic partial differential equation (SPDE). Examples are found in a broad range of different applications, including surface growth dynamics such as e.g. surface erosion by ion sputtering processes [1,2], film deposition in electrochemistry [3,4], or by other methods [5,6], fluid flow in porous media [7–9], fracture dynamics [10] and thin film dynamics [11–15], to name but a few. Not surprisingly, understanding the dynamics of the fluctuating interface in terms of its roughening properties, which often exhibit scale-invariant universal features and long-range spatiotemporal correlations, has become an important problem in statistical physics which has received considerable attention over the last decades [16]. In addition, the ability of controlling not only the dynamics of the surface roughness (e.g. its growth rate) but also its convergence towards a desired saturated value has recently received an increased interest due to its applicability in a wide spectrum of natural phenomena and technological applications.

Here we present a generic linear control methodology for controlling the surface roughness, i.e., the variance of the solution, of nonlinear SPDEs which we exemplify with the stochastic Kuramoto–Sivashinsky (sKS) equation. The starting point is to split the original SPDE into a stochastic linear part and a deterministic nonlinear part, and to apply existing control methodologies [17,18] to the nonlinear deterministic part. Our control strategy is based on two steps: first, stabilize the zero solution of the deterministic system and, second, control the second moment of the solution of the stochastic linear equation (e.g. a measure of the surface

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roughness) to evolve towards any desired value. By considering either periodic or point actuated controls, our results show that the second moment of the solution grows in time according to a power-law with a well-defined growth exponent until it saturates to the prescribed value we wish to achieve.

It is important to note that other control strategies have been proposed previously for controlling the surface roughness and other quantities of interest, such as the film porosity and film thickness in various linear dissipative models, including the stochastic heat equation, the linear sKS equation, and the Edwards–Wilkinson (EW) equation; see e.g. [5,6,19–25]. However, it should also be emphasized that most of these works involve the use of nonlinear feedback controls which change the dynamics of the system and require knowledge of the nonlinearity at all times, something that may be difficult to achieve. We believe that our framework offers several distinct advantages since the controls we derive and use are linear functions of the solution which do not affect the overall dynamics of the system and also decrease the computational cost. Another recent study is Ref. [26] which considered a deterministic version of the KS equation, and presented a numerical study of the effects of the use of ion bombardment which varies periodically in time on the patterns induced by the ion beams on an amorphous material. In particular, this study found that rocking the material sample about an axis orthogonal to the surface normal and the incident ion beam, which corresponds to making the coefficients of the KS equation periodic in time, can lead to suppression of spatiotemporal chaos.

The work presented in this paper is motivated by earlier research carried out by our group: on one hand, the study of noise induced stabilization for the KS equation [27,28] and, on the other hand, the study of optimal and feedback control methodologies for the KS equation and related equations that are used in the modeling of falling liquid films [17,18,29]. It was shown in [27,28] that an appropriately chosen noise can be used in order to suppress linear instabilities in the KS equation, close to the instability threshold. Furthermore, it was shown in [17,18] that nontrivial steady states and unstable traveling wave solutions of the deterministic KS equation can be stabilized using appropriate optimal and feedback control methodologies. In addition, similar feedback control methodologies can be used in order to stabilize unstable solutions of related PDEs used in the modeling of falling liquid films, such the Benney and weighted-residuals equations.

The rest of the paper is structured as follows. Section 2 introduces the sKS equation and discusses means to characterize the roughening process of its solution. In Section 3 we outline the general linear control methodology which is applied to the case of periodic controls in Section 4, and point actuated controls in Section 5. A summary and conclusions are offered in Section 6.

2. The stochastic Kuramoto–Sivashinsky (sKS) equation

Consider the sKS equation:

$$u_t = -\nu u_{xxxx} - u_{xx} - uu_x + \sigma\xi(x,t), \tag{1}$$

normalized to 2π domains ($x \in [0, 2\pi]$) with $\nu = (2\pi/L)^2 > 0$, where L is the size of the system, with periodic boundary conditions (PBCs) and initial condition $u(x,0) = \phi(x)$. $\xi(x,t)$ denotes Gaussian mean-zero spatiotemporal noise, which is taken to be white in time, and whose strength is controlled by the parameter σ:

$$\left\langle \xi(x,t)\xi(x',t')\right\rangle = \mathcal{G}(x-x')\delta(t-t'), \tag{2}$$

where $\mathcal{G}(x-x')$ represents its spatial correlation function. We can, in principle, consider the control problem for SPDEs of the form (1) driven by noise that is colored in both space and time. Such a noise can be described using a linear SPDE (Ornstein–Uhlenbeck process) [30].

The noise term can be expressed in terms of its Fourier components as:

$$\xi(x,t) = \sum_{k=-\infty}^{\infty} q_k \dot{W}_k(t)\, e^{ikx}, \tag{3}$$

where $\dot{W}_k(t)$ is a Gaussian white noise in time and the coefficients q_k are the eigenfunctions of the covariance operator of the noise. For example, if $\mathcal{G}(x-x') = \delta(x-x')$ (which corresponds to space–time white noise), we have $q_k = 1$. For the noise to be real-valued, we require that the coefficients q_k verify $q_{-k} = q_k$. Proofs of existence and uniqueness of solutions to Eq. (1) can be found in [31,32], for example. The behavior of Eq. (1) as a function of the noise strength, and for particular choices of the coefficients $\{q_k\}$ has been analyzed in detail in [27,28]. In particular, it was shown that sKS solutions undergo several state transitions as the noise strength increases, including critical on–off intermittency and stabilized states.

The quadratic nonlinearity in Eq. (1) is typically referred to as a Burgers nonlinearity. We note that an alternative version of Eq. (1) is found by making the change of variable $u = -h_x$, giving rise to

$$h_t = -\nu h_{xxxx} - h_{xx} + \frac{1}{2}(h_x)^2 + \sigma\eta(x,t), \tag{4}$$

where $\xi(x,t) = \partial_x\eta(x,t)$. The main effect of this transformation is to change the dynamics of the average $u_0(t) = \frac{1}{2\pi}\int_0^{2\pi} u(x,t)\,dx$ of the solution. Indeed, Eq. (1) with PBCs preserves the value of u_0 whereas as a consequence of the nonlinear term $(h_x)^2$, Eq. (4) does not conserve the mass $h_0(t) = \frac{1}{2\pi}\int_0^{2\pi} h(x,t)\,dx$. Both equations have received a lot of attention over the last decades, with Eq. (1) more appropriate in mass-conserved systems such as the dynamics of thin liquid films [28,11–15], and Eq. (4) relevant in modeling surface growth processes such as surface erosion by ion sputtering processes [3,4,1,2,33,22,34]. It is also worth mentioning that the quadratic nonlinearity appearing in Eq. (4) is the same as that in the Kardar–Parisi–Zhang (KPZ) equation [35,36]

$$h_t = h_{xx} + \frac{1}{2}(h_x)^2 + \sigma\eta(x,t). \tag{5}$$

In fact extensive work indicates that Eqs. (4) and (5) are asymptotically equivalent, something referred to as the “Yakhot conjecture” [37–39]. Throughout the remainder of this study we will refer to Eq. (1) as the sKS equation with Burgers nonlinearity and Eq. (4) as the sKS equation with KPZ nonlinearity.

2.1. Surface roughening

An important feature of systems involving dynamics of rough surfaces is that one often observes the emergence of scale invariance both in time and space, i.e., the statistical properties of quantities of interest are described in terms of algebraic functions of the form $f(t) \sim t^\beta$ or $g(x) \sim x^\alpha$, where α and β are referred to as scaling exponents. An example of this is the surface roughness, or variance of $u(x,t)$, which is defined as

$$r(t) = \sqrt{\frac{1}{2\pi}\int_0^{2\pi} \left[u(x,t) - u_0(t)\right]^2\, dx}. \tag{6}$$

We remark that u_0 may or may not depend on time, depending on whether we consider the Burgers or the KPZ nonlinearities. Usually the above quantity grows in time until it reaches a saturated regime, in which the fluctuations become statistically independent