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The stability spectrum for elliptic solutions to the sine-Gordon equation

Bernard Deconinck [a], Peter McGill [b], Benjamin L. Segal [a,*]

[a] Department of Applied Mathematics, University of Washington, Seattle, WA 98195-3925, USA
[b] Department of Physics and Astronomy, University College London, London, WC1E 6BT, UK

HIGHLIGHTS

- A complete characterization of the stability spectrum for stationary elliptic-type solutions to the sine-Gordon equation.
- A classification of the stability of solutions with respect to subharmonic perturbations.
- An explicit description of the spectrum for a family of non-self adjoint problems.

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ABSTRACT

We present an analysis of the stability spectrum for all stationary periodic solutions to the sine-Gordon equation. An analytical expression for the spectrum is given. From this expression, various quantitative and qualitative results about the spectrum are derived. Specifically, the solution parameter space is shown to be split into regions of distinct qualitative behavior of the spectrum, in one of which the solutions are stable. Additional results on the spectral stability of solutions with respect to perturbations of an integer multiple of the solution period are given.

1. Introduction

The sine-Gordon equation in laboratory coordinates is given by

$$u_{tt} - u_{xx} + \sin u = 0. \tag{1}$$

Here, $u(x,t)$ is a real-valued function. This equation was first introduced to study surfaces of constant Gaussian curvature in light cone form [1]. Since its introduction it has appeared in various applications including the description of the magnetic flux in long superconducting Josephson junctions [2–4], the modeling of fermions in the Thirring model [5], the study of the stability of structures found in galaxies [6–8], mechanical vibrations of a ribbon pendulum [9], propagation of crystal dislocation [10], propagation of deformations along DNA double helix [11], among others. A comprehensive discussion of many of these applications is found in the review paper by Barone [12].

We consider general traveling wave solutions to (1). Defining $z = x - ct$, $\tau = t$, and introducing $v(z,\tau) = u(x,t)$,

$$(c^2 - 1)v_{zz} - 2c v_{z\tau} + v_{\tau\tau} + \sin(v) = 0. \tag{2}$$

For subsequent discussion we assume that $c \neq 1$. We proceed to look for stationary solutions to (2) of the form

$$v(z,\tau) = f(z), \tag{3}$$

* Corresponding author.
E-mail address: bsegal@uw.edu (B.L. Segal).

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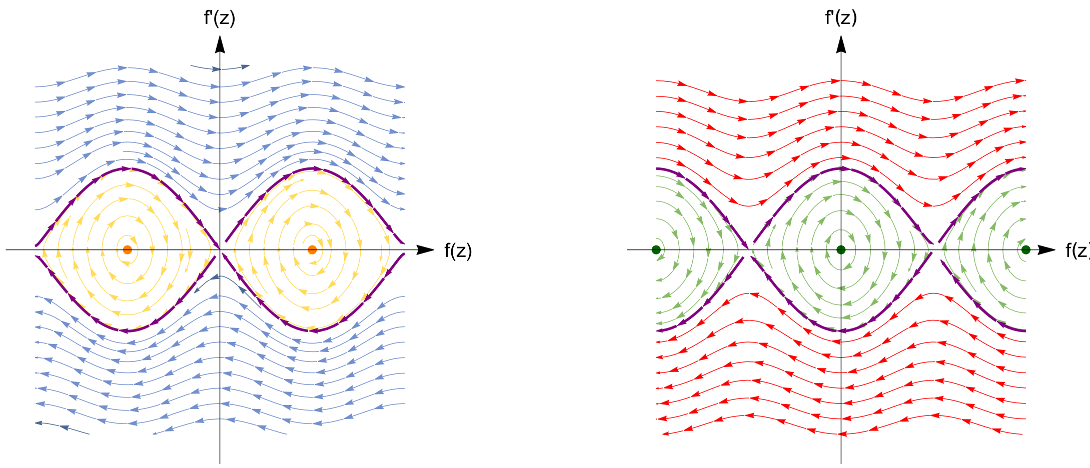

(a) Subluminal: $c^2 < 1$.

(b) Superluminal: $c^2 > 1$.

Fig. 1. Phase portraits of the solutions showing both librational waves (closed orbits inside the separatrix) in yellow for (a) and green for (b) and rotational waves (orbits outside the separatrix) in blue for (a) and red for (b). The separatrix is denoted in purple. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

leading to

$$(c^2-1)f''(z)+\sin(f(z))=0, \tag{4}$$

where $'$ denotes a derivative with respect to z. Integrating once,

$$\frac{1}{2}(c^2-1)f'(z)^2+1-\cos(f(z))=E, \tag{5}$$

where E is a constant of integration referred to as the total energy. The stationary solutions in this paper are the elliptic solutions to (5) and their limits. These solutions are periodic in z and limit to the well-known kink solutions as their period goes to infinity [13,14].

We call stationary solutions $f(z)$ with waves speeds satisfying $c^2 < 1$ (respectively $c^2 > 1$) subluminal (superluminal). Representative phase portraits of subluminal and superluminal solutions to (5) are shown in Fig. 1. Additionally, we call solutions $f(z)$ whose orbits in phase space lie within the separatrix librational, and those whose orbits lie outside the separatrix rotational. This distinction is illustrated in Fig. 1 in both the subluminal and superluminal cases. Librational waves correspond to $E \in (0, 2)$. For rotational waves, $E < 0$ for subluminal waves and $E > 2$ for superluminal waves.

Scott [15] was the first to study the stability of periodic traveling wave solutions to (1). He classified subluminal rotational waves as spectrally stable and determined spectral instability for all other types of waves, but these instability results were based on an incorrect claim that the spectrum in all cases was strictly confined to the real and imaginary axes. His proof has been corrected [16] and extended to the Klein–Gordon equation [17]. Using entirely different methods, we confirm the results in [16] and explicitly characterize all of parameter space. We also provide stability results for solutions perturbed by integer multiples of their fundamental period.

In Section 2 we present the elliptic solutions to (5) in Jacobi elliptic form from [16], and then reformulate the solutions into Weierstrass elliptic form. In Sections 3–5, using the same methods as [18–21], we exploit the integrability of (1) to associate the spectrum of the linear stability problem with the Lax spectrum using the squared eigenfunction connection [22]. This allows us to obtain an analytical expression for the spectrum of the operator associated with the linearization of (1) in the form of a condition on the real part of an integral over one period of some integrand. Similar to [21], we proceed by integrating the integrand explicitly in Section 6. Next, using the expressions obtained, we prove results concerning the location of the stability spectrum on the imaginary axis in Section 7. In Section 8, we present analytical results about the spectrum, and we make use of the integral condition to split parameter space into different regions where the spectrum shows qualitatively different behavior. Finally, in Section 9 we examine the spectral stability of solutions with respect to perturbations of an integer multiple of their fundamental period and prove various stability results.

2. Elliptic solutions

The derivation of the solutions is presented in the appendix of [16]. We limit our presentation to what is necessary for the following sections. For solutions to be real and nonsingular for real z we require the following constraints:

subluminal, rotational:	$0 \le \lvert c\rvert < 1,$	$E < 0,$	(6)
superluminal, rotational:	$\lvert c\rvert > 1,$	$E > 2,$	(7)
subluminal, librational:	$0 \le \lvert c\rvert < 1,$	$0 < E \le 2,$	(8)
superluminal, librational:	$\lvert c\rvert > 1,$	$0 < E \le 2.$	(9)

Solutions to (5) are of the form

$$\cos(f(z))=\alpha+\beta\,\mathrm{sn}^2(\lambda z,k), \tag{10}$$