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Oscillatory instabilities of gap solitons in a repulsive Bose–Einstein condensate

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HIGHLIGHTS

- We study numerically the oscillatory instabilities of fundamental gap solitons.
- We use the Evans function combined with the exterior algebra formulation.
- The case of repulsive interactions is considered.
- The oscillatory instability found for fundamental gap solitons in the first and second gaps.
- The results are confirmed by direct simulations of the dynamics.

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ABSTRACT

The paper is devoted to numerical study of stability of nonlinear localized modes ("gap solitons") for the spatially one-dimensional Gross–Pitaevskii equation (1D GPE) with periodic potential and repulsive interparticle interactions. We use the Evans function approach combined with the exterior algebra formulation in order to detect and describe weak oscillatory instabilities. We show that the simplest ("fundamental") gap solitons in the first and in the second spectral gap undergo oscillatory instabilities for certain values of the frequency parameter (i.e., the chemical potential). The number of unstable eigenvalues and the associated instability rates are described. Several stable and unstable more complex (non-fundamental) gap solitons are also discussed. The results obtained from the Evans function approach are independently confirmed using the direct numerical integration of the GPE.

1. Introduction

The Gross–Pitaevskii equation (GPE),

$$i\Psi_t = -\Psi_{xx} + V(x)\Psi + \sigma|\Psi|^2\Psi, \tag{1}$$

describes the meanfield dynamics of a quasi-one-dimensional Bose–Einstein condensate (BEC) confined in the potential $V(x)$ [1]. In Eq. (1), $\Psi = \Psi(t,x)$ is the complex-valued macroscopic wavefunction of the condensate. The squared amplitude of the wavefunction $|\Psi(t,x)|^2$ describes the local density of the BEC, while the gradient $(\arg\Psi(t,x))_x$ describes the velocity of atoms of condensate. The nonlinear term $\sigma|\Psi|^2\Psi$ takes into account the interactions between the particles. The case $\sigma = 1$ corresponds to repulsive interparticle interactions, while $\sigma = -1$ describes attractive interactions. Both these cases are of physical relevance, but in what follows, we mainly focus on the *repulsive case*, $\sigma = 1$. It is assumed that the potential $V(x)$ is *periodic*, which corresponds to the optical confinement of the BEC [2,3].

An important class of solutions of the GPE (1) corresponds to the stationary modes which can be represented in the form $\Psi(t,x) = e^{-i\mu t}u(x)$, where μ is a real parameter having the meaning of the chemical potential of the BEC. Function $u(x)$ satisfies the conditions of the spatial localization

$$\lim_{x\to\pm\infty} u(x) = 0. \tag{2}$$

Without loss of generality one can assume that $u(x)$ is real-valued [4]. Then $u(x)$ can be found from the stationary GPE

$$u_{xx} + (\mu - V(x))u - \sigma u^3 = 0. \tag{3}$$

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If the potential $V(x)$ is periodic, the nonlinear modes satisfying (2)–(3) are called *gap solitons* [2,3,5–8], since values of μ corresponding to these solutions lie in the spectral gaps of the linear Schrödinger equation [9]

$$u_{xx} + (\mu - V(x))u = 0. \tag{4}$$

The simplest class of the gap solitons are *fundamental gap solitons* (FGSs) [5,7,10–12]. Under the repulsive nonlinearity, in the first gap there exists one family of FGSs which are characterized by the presence of a single dominating peak localized in one well of the potential $V(x)$. A variety of more complex (or *higher-order*) solitons includes truncated Bloch waves [13,14] (which consist of several in-phase peaks placed in a row), various asymmetric states, and complex bound states of two (or more) well-separated waves [15], etc. In spite of their rich diversity, under certain (not very restrictive) conditions *all* possible gap solitons in a repulsive BEC can be viewed as complexes of FGSs and classified using an alphabet consisting of a few symbols [16]. Specifically, if the lattice is deep enough, then all the gap solitons in the first gap can be put into a one-to-one correspondence with the set of bi-infinite sequences of symbols from a three-symbol alphabet. In simple terms, these symbols denote the presence or the absence of the FGS (taken with plus or minus sign) in a potential well situated on the period of the potential $V(x)$. For instance, the truncated Bloch waves [13,14] consisting of several in-phase peaks placed in a row can be viewed as complexes of single-hump FGSs. For classification of the gap solitons in the second spectral gap, an alphabet of five symbols is necessary, and so on.

An important property of a gap soliton is its stability, since only dynamically stable modes are likely to be experimentally feasible. The literature about stability of gap solitons is rather abundant [6–8,11–13,17–19]. The most relevant for our study outcomes for the case of repulsive interactions ($\sigma = 1$) can be summarized as follows. Significant part of the studies concluded that the single-hump FGSs are stable in the first gap [6,7,10–12] and in the second gap [10–12]. Regarding the higher-order states consisting of two or three in-phase peaks, they have been reported unstable near the upper band edge in [7] in the first gap. However, these states have been found to be stable both in the first [11,13] and in the second [11] gap if the lattice depth is large enough.

The results listed above have been obtained on the basis of numerical studies of stability. In the meanwhile, it is recognized that the numerical analysis of stability of the gap solitons is quite a delicate problem. A standard approach to the stability relies on the linear (or spectral) stability technique which reduces the stability question to a study of the spectrum of a certain linear operator. Depending on the character of unstable eigenvalues, the instability typically manifests itself either as a purely *exponential instability* (when the unstable eigenvalues have zero imaginary parts) or as an *oscillatory instability* (OI) (when the unstable eigenvalues are complex with nonzero imaginary parts). While the instabilities of the former type are relatively simple to detect [8,18], the accurate tracing of OIs is much more challenging [7,8]. As a result, the information about OIs of gap solitons in optical lattices is rather scarce. The absence of information on OIs for the simplest one-hump gap solitons in GPE is especially remarkable in view of well-known OIs of Bragg gap solitons in nonlinear Dirac equations [20–22]. The latter system can be deduced from the GPE with a shallow periodic potential using an asymptotic multiple-scale expansion [17,23], and therefore the results about OIs of the solitons for Dirac system seem to be not consistent with the stability results for the single-hump FGS mentioned above.

The numerical difficulties arising in the analysis of the OIs of the gap solitons are related to several issues. First, the rates of OIs are typically quite small [8,18]. Another difficulty results from poor localization of the gap soliton and (or) of the eigenfunction associated with an unstable eigenvalue. This situation typically takes place when the chemical potential μ is close to the gap edge. It requires unpractically wide computational windows or a particularly accurate treatment of the boundary conditions. Some of these difficulties can be overcome using the Evans function approach which was employed in [8] to trace OIs of gap solitons in the attractive condensates. It was further demonstrated in [24] that the numerically accurate evaluation of the Evans function requires a careful treatment of the stiffness issue which arises for some values of the complex argument of the Evans function. The stiffness problem can be fixed if one redefined the Evans function using the exterior algebra formalism [22,24]. This idea has been developed into a robust numerical technique which was demonstrated to provide reliable results even for relatively weak instabilities of gap solitons [22,24].

In the present paper, we use the Evans function approach combined with the exterior algebra formulation in order to reveal and describe weak OIs of FGS and higher-order gap solitons in the repulsive BEC. We focus on the first and second spectral gaps. In each gap, we consider the single-hump FGS and two higher-order solitons bearing two or three in-phase humps. The main outcomes of our numerical study can be outlined as follows.

1. In the first gap, all the considered solitons (including the single-hump FGS) are stable far from the upper band edge, but undergo OIs in the region near the upper band edge. The width of this instability region is quite significant: it occupies about 15%–20% of the width of the first gap.
2. In the second gap, all the considered solitons (including the FGSs) are, in general, unstable due to OIs. However, in a sufficiently deep potential, there exist intervals of μ where FGS are stable.

To the best of our knowledge, our results constitute the first explicit demonstration and detailed description of OIs for FGSs in the case of repulsive interactions $\sigma = 1$ in the first and in the second gap. On the other hand, our results advance the current understanding of the higher-order modes [13], since we show that they undergo OIs even in a deep potential.

In order to confirm the linear stability results, we have also performed a series of direct simulations of temporal behavior of the solitons in the GPE (1). The results of these studies agree with the conclusions obtained from the linear stability analysis and display the slow decay of unstable gap solitons and the persistent evolution of the stable ones.

The rest of the paper is organized as follows. In Section 2 we briefly describe the families of gap solitons whose stability is the main subject of the present study. In Section 3 we present our main results on linear and nonlinear stability of the gap solitons. Section 4 concludes the paper.

2. Families of gap solitons

In our study, we use a prototypical example of the periodic potential in the form

$$V(x) = -V_0 \cos(2x), \tag{5}$$

where real $V_0 > 0$ characterizes the depth of the lattice.

The spectrum of the linear eigenvalue problem (4) with the potential (5) consists of one semi-infinite gap and a countable set of finite gaps [9] (see Fig. 1). In our study, we consider the gap solitons in the first and in the second gap (no gap solitons exist in the semi-infinite gap in repulsive BECs). In each of the gaps, we consider three families of nonlinear modes: the fundamental gap soliton with the single dominating peak at $x = 0$ (the single-hump FGS), and two higher-order solitons, with two and three dominating peaks. In the terminology of [13], these higher-order