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Limit cycles in planar piecewise linear differential systems with nonregular separation line

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HIGHLIGHTS

- The nonregularity in the separation line occurs only in one point.
- The number of periodic orbits is bigger than for the regular case.
- All the periodic orbits have the breaking point in its interior.
- Higher Melnikov theory is used for the described bifurcating phenomena.
- The stabilization phenomena in the number appear increasing the order.

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ABSTRACT

In this paper we deal with planar piecewise linear differential systems defined in two zones. We consider the case when the two linear zones are angular sectors of angles α and $2\pi-\alpha$, respectively, for $\alpha \in (0,\pi)$. We study the problem of determining lower bounds for the number of isolated periodic orbits in such systems using Melnikov functions. These limit cycles appear studying higher order piecewise linear perturbations of a linear center. It is proved that the maximum number of limit cycles that can appear up to a sixth order perturbation is five. Moreover, for these values of α, we prove the existence of systems with four limit cycles up to fifth order and, for $\alpha=\pi/2$, we provide an explicit example with five up to sixth order. In general, the nonregular separation line increases the number of periodic orbits in comparison with the case where the two zones are separated by a straight line.

1. Introduction

Many systems of relevance to applications are modeled using piecewise linear differential systems. The study of such systems goes back to Andronov and coworkers [1] and nowadays still continues receiving attention by many researchers. For more details about piecewise linear (and piecewise smooth in general) differential systems see for instance the books of Filippov [2] and di Bernardo et al. [3] and the references quoted therein.

In the classical theory for smooth systems an important topic is the weak 16th Hilbert's problem. The question is: Which is the maximum number of isolated periodic orbits, also called limit cycles, that bifurcate perturbing a center? This problem for piecewise differential systems defined in two zones have been studied recently, among other papers, in [4–12]. Usually the separation line between the two zones is a straight line. Here we study the case when the separation line is nonregular. angular regions, i.e. the separation line is formed by two semi-straight lines that coincide at the origin forming an angle α, with $\alpha \in (0,\pi)$. In particular we provide lower bounds for the number of limit cycles of the linear center under perturbation, with piecewise linear vector fields, up to order six. After a linear transformation, if it is necessary, it is not restrictive to assume that the center is the classic harmonic oscillator. More precisely, for each

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$N \in \mathbb{N}$, we consider the following piecewise linear perturbation of the linear center

$$\begin{cases} \dot{x} = -y + \sum_{i=1}^{N} \varepsilon^i (a_{0i}^{\pm} + a_{1i}^{\pm}x + a_{2i}^{\pm}y), \\ \dot{y} = x + \sum_{i=1}^{N} \varepsilon^i (b_{0i}^{\pm} + b_{1i}^{\pm}x + b_{2i}^{\pm}y), \end{cases} \quad (1)$$

defined in the angular regions separated by the line Σ_α. In fact the separation line Σ_α is defined as follows. For $\alpha \in (0, \pi)$ and $\alpha \neq \pi/2$, then $\Sigma_\alpha = \{(x, y) : x \geq 0, y = 0\} \cup \{(x, y) : x = (\tan\alpha)^{-1}y, y \geq 0\}$. For $\alpha = \pi/2$, we have $\Sigma_{\frac{\pi}{2}} = \{(x, y) : x \geq 0, y = 0\} \cup \{(x, y) : x = 0, y \geq 0\}$. Finally, Σ_π denotes the straight line $\{(x, y) : y = 0\}$. The notations $\Sigma_\alpha^{\pm}$ indicate the angular sectors of angles α and $2\pi - \alpha$ separated by Σ_α, respectively. We denote the vector fields associated to system (1), defined in $\Sigma_\alpha^{\pm}$, by $X^{\pm}$, respectively. The point $(0, 0)$ where the separation line Σ_α loses its regularity will be referred to as the *breaking point*.

It is worth to emphasize that, with the perturbations that we have considered, the perturbed systems do not escape from the class of piecewise linear system, but we consider the period annulus of the center instead of a neighborhood of the origin. This is the aim of the higher order Poincaré–Pontryagin–Melnikov theory instead of degenerated Hopf bifurcation. This theory provides the same results, in the plane, than the averaging one. In this paper, N denotes the degree in the perturbation parameter ε, or the order of perturbation in ε.

The number of limit cycles close to the origin for piecewise families, using Lyapunov constants, is studied in [13,12]. All the families introduced in both works have the origin as a critical point for the systems defined in $\Sigma_\pi^{\pm}$, respectively. In fact, the perturbations are of higher order in the variables. In our case the perturbations are linear in the variables but nonlinear in the parameter ε. Moreover, we do not preserve the origin as a critical point in $\Sigma_\alpha^{\pm}$, then the technique used in those papers, based on a change to polar coordinates, is more difficult to apply. Consequently these two problems are not equivalent.

In the case when the separation line is a straight line, Han and Zhang in [7] conjectured that the maximum number of limit cycles for planar discontinuous piecewise linear systems should be at most two. However, Huan and Yang in [8] provided strong numerical evidence that three limit cycles should exist. A computer-assisted proof of the existence of such limit cycles was given in [10]. The existence of other examples with three limit cycles, via bifurcation techniques, can be found in [4,14]. The example given in [4] uses a piecewise linear perturbation of a linear center and it is proved that three is the maximum number of limit cycles that can appear up to a seventh order perturbation. Moreover, as was observed in [4], when the order of the perturbation increases, the number of limit cycles seems to stabilize in three. However, it is still an open question to determine whether three is the maximum number of limit cycles for planar discontinuous piecewise linear differential systems when the separation line is a straight line. In this case, Euzébio and Llibre in [15] proved that if one of the linear differential systems has its equilibrium point on the straight line of separation, then the maximum number is less than or equal to four. This upper bound is decreased by two in the same cases in [16,17]. For this special class, the complete study is done in [18], where it is shown that the maximum number of limit cycles is two. Moreover, this upper bound is reached.

When the separation line is no longer a straight line, it is possible to obtain more than three limit cycles. Braga and Mello in [19] showed the importance of the separation boundary in the number of bifurcated limit cycles. They proved the existence of piecewise linear differential systems with two zones in the plane with four, five, six and seven limit cycles, and conjectured that, given $n \in \mathbb{N}$, there is a piecewise linear system with two zones in the plane with exactly n limit cycles. Promptly, Novaes and Ponce in [20] gave a positive answer to this conjecture. Braga and Mello in [21] also showed the existence of a class of discontinuous piecewise linear differential systems with two zones in the plane having exactly n hyperbolic limit cycles. As it was pointed out in [21], in the obtained examples in [20], the limit cycles can be nonhyperbolic.

In this article, we highlight once again the importance of the separation line and the number of breaking points in the number of limit cycles that can appear by perturbation in piecewise linear vector fields. We study the bifurcation of limit cycles by studying higher order piecewise linear perturbations of a linear center. We follow the procedure described in [4] to study the Σ_α-piecewise linear vector field and we get four limit cycles for every $\alpha \in (0, \pi)$ and five for $\alpha = \pi/2$. This shows that, in general, one can obtain more limit cycles in comparison with the case of Σ_π-piecewise linear vector fields. Clearly, the functions to be studied, which ensure the existence of all these limit cycles, cannot be well defined when α goes to π or to 0. We will come back to this question later.

In the works [19,20] as well as in the present paper, all the considered limit cycles are nested and they intersect Σ_α only at crossing points. That is, the limit cycles intersecting the sliding region are not considered. Below we give the precise definitions. Besides the method used, the main qualitative difference with [19] is that we have only one breaking point which defines the nonregular set in Σ_α, and not one between two consecutive limit cycles. Moreover all our limit cycles have this breaking point in its interior. With respect to [20] we observe that the separation line is analytic.

For analytic vector fields the number of limit cycles usually increases when higher order perturbations are considered. It is well known that, up to a first order analysis in ε, perturbing the linear center with arbitrary polynomials of degree n, we can only obtain $[(n-1)/2]$ limit cycles for the perturbed system, where $[\cdot]$ denotes the integer part function, see [22]. On the other hand but in the same class of systems, in [23] it is proved that the maximum number of limit cycles is lower than or equal to $[N(n-1)/2]$. This upper bound, in general, is reached when n is large enough and $N = 2$. In many classes of polynomial systems, when N increases, the number of limit cycles usually stabilizes. The stabilization process depends on the considered family. In [24] this phenomenon is studied for some families. For example, a concrete class is presented such that the maximum number of limit cycles is 0, 0, 1, 1, 1, 2, 2, 2, 2, 2 when $N = 1, \ldots, 10$. In [23], considering perturbations of a linear center by quadratic polynomials, it is shown that when $N = 1, \ldots, 6$, the maximum number of limit cycles is 0, 1, 1, 2, 2, 3, respectively. A higher order study is not necessary because Bautin in [25], for quadratic systems, proves that at most three limit cycles can appear near a focus or a center. This stabilization phenomenon also appears in piecewise linear systems. In [4] it is proved that for system (1), with separation line Σ_π, the maximum number of limit cycles is 1, 1, 2, 3, 3, 3, 3 when $N = 1, 2, 3, 4, 5, 6, 7$, respectively. In this paper we have not shown if the stabilization procedure also appears in general piecewise linear systems with nonregular separation line because of the computations. But we think that this phenomenon will appear for every family of systems, as we show in Section 5 for some classes of Σ_α-piecewise linear Liénard systems.