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Coupled oscillators on evolving networks

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HIGHLIGHTS

- Mechanism for collective phenomena on stochastically evolving networks is proposed.
- Oscillator systems exhibit synchronization and amplitude death as steady states.
- Nonlinearity controls the time to reach the steady state and its nature.
- Results are independent of the transition rules for stochastic evolution.

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ABSTRACT

In this work we study coupled oscillators on evolving networks. We find that the steady state behavior of the system is governed by the relative values of the spread in natural frequencies and the global coupling strength. For coupling strong in comparison to the spread in frequencies, the system of oscillators synchronize and when coupling strength and spread in frequencies are large, a phenomenon similar to amplitude death is observed. The network evolution provides a mechanism to build inter-oscillator connections and once a dynamic equilibrium is achieved, oscillators evolve according to their local interactions. We also find that the steady state properties change by the presence of additional time scales. We demonstrate these results based on numerical calculations studying dynamical evolution of limit-cycle and van der Pol oscillators.

1. Introduction

Synchronization in a system of coupled oscillators has been investigated intensely in the past and it still continues to be an active field of research due to its potential implications in various natural and artificial systems [1,2]. The Kuramoto model [3,4] is one of the most extensively studied models of synchronization in a system of coupled oscillators [5]. Being a model of phase oscillators, the Kuramoto model does not exhibit features such as amplitude death [6], which has been observed in a system of coupled limit-cycle oscillators [7], chimera states, where synchronization and asynchronization amongst the oscillators coexist [8,9]. These properties are not merely solutions of the dynamical equations that describes coupled oscillator system, but also manifest in realistic examples, e.g., coupled electrochemical oscillators [10]. Amplitude death arises when the inter-oscillator coupling becomes very strong in comparison to their limit-cycle attractions and this transition takes place via a Hopf bifurcation [11]. The phenomena of amplitude death and oscillator synchronization has also been observed in a large population of limit-cycle oscillators [12] and in various nonlinear systems on different coupling topologies [13,14]. The studies on coupled oscillator systems have mainly focused on static interaction topologies, whereas in most of the natural systems the interaction between components are observed to change with time [15,16].

In a recent work on a system of Kuramoto oscillators on an evolving network it was shown that the phase oscillators synchronize for strong enough coupling strength in a network which is constantly evolving in time [17]. This collective behavior of the coupled oscillator system is robust against local temporal fluctuations of the network topology in steady state. Such an autonomous evolution of the system provides a mechanism for the self-organized behavior where network topology and oscillator dynamics coevolve in an interdependent way. This cumulative effect of topology and dynamics is a ubiquitous property of many natural systems and the mechanism proposed successfully captures this for phase oscillator systems. However, general oscillator systems also have fluctuations in their amplitudes and a model

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based on phase oscillators does not capture this feature. In this work we study a system of coupled phase–amplitude oscillators on an evolving network to investigate the role of temporal fluctuations in oscillator couplings. Here, we consider nonlinear oscillators and study the effect of nonlinearity on global synchronization. The network topology evolves stochastically and depends on both phase and amplitude of the oscillators. In Section 2 we present a general formulation of the model for a complex dynamical system. In Section 3 we consider a system of linearly coupled limit-cycle and van der Pol oscillators on evolving networks followed by the set of transition rules governing the evolution of interaction topology in accordance with the oscillator and network degrees of freedom. Finally, results are presented in Section 4.

2. The model

A complex dynamical system typically consists of a collection of individual units. The system evolves collectively depending only on the nature of interaction between individual units. The state of the complex dynamical system is defined by the vector $\mathbf{x} = (x_1, \ldots, x_N)$ where x_i denotes the dynamical variable for the ith unit along with the adjacency matrix $\mathbf{g}$ that represents the interaction i.e. links between individual units. The evolution of the dynamical system is given by

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{g}, t), \tag{1}$$

for all $t \in [t_n, t_{n+1})$, $n = 0, 1, 2, \ldots$ with the initial conditions $\mathbf{x}(t_0) = \mathbf{x}_0$, and $\mathbf{g}(t_0) = \mathbf{g}_0$. By virtue of the continuity of the solution at t_n, we require, $\lim_{t\to t_n} \mathbf{x}(t) = \mathbf{x}(t_n)$. The adjacency matrix $\mathbf{g}$ evolves stochastically at discrete time steps $t = t_1, t_2, t_3, \ldots$ and can be written as

$$\mathbf{g}(t_{n+1}) = \mathbf{g}' \quad \in \Gamma \quad \text{with probability } P(\mathbf{g}'|\mathbf{g}(t_n); \mathbf{x}(t_{n+1})), \tag{2}$$

where Γ is the set of all networks having N nodes. Note that Eq. (2) defines a Markov process [18] and successive adjacency matrices $\mathbf{g}(t_n)$ and $\mathbf{g}(t_{n+1})$ differ only by a single link. This is due to the fact that in every interval $[t_n, t_{n+1})$, $n = 0, 1, \ldots$ there is only one or less element of the adjacency matrix is changed by Eq. (2) at time $t = t_{n+1}$. The discrete evolution of the network is schematically described in Fig. 1.

The transition probability P in Eq. (2) depends on $\mathbf{x}(t_{n+1})$ which can be written as a function of $\mathbf{g}(t_n)$ and $\mathbf{x}(t_n)$ by integrating Eq. (1) in $[t_n, t_{n+1})$. The transition probability satisfies $0 \le P(\mathbf{g}'|\mathbf{g}(t_n); \mathbf{x}(t_{n+1})) < 1$ and $\sum_{\mathbf{g}'\in\Gamma} P(\mathbf{g}'|\mathbf{g}(t_n); \mathbf{x}(t_{n+1})) = 1$. Furthermore, we assume that the transition probability is separable in $\mathbf{x}$ and $\mathbf{g}$ which leads to $P(\mathbf{g}'|\mathbf{g}(t_n); \mathbf{x}(t_{n+1})) = \eta(\mathbf{g}'|\mathbf{g}(t_n))\rho(\mathbf{x}(t_{n+1}))$ for some functions η and ρ. The probability that a link is formed or broken at time t_n between two individual units depend on the number of incoming and outgoing links of the pair of nodes. It should also depend on the dynamical variable $\mathbf{x}(t_n)$. However, for different values of $\mathbf{x}(t_n)$ i.e., dynamical states of the system, if we change the topology of the network without changing the number of incoming and outgoing links at each node, the probability should be proportional only to some function i.e., $\rho(\mathbf{x}(t_{n+1}))$, of the dynamical variable. Therefore we assume that the transition probability is a product of the two functions $\eta(\mathbf{g}'|\mathbf{g}(t_n))$ and $\rho(\mathbf{x}(t_{n+1}))$. The exact form of the transition probability for the case of coupled oscillators is discussed in Section 3.

The interdependent evolution of the dynamical variables and the network topology described by Eqs. (1) and (2) can give rise to cooperative behavior provided Eq. (1) has a steady state solution as the network topology approaches a certain stationary topology for a given transition rule. Often, cooperative behavior of the system may be characterized by an order parameter r which is a scalar function of the dynamical variables and can be defined as $r = \Phi(\mathbf{x})$. The order parameter r attains a stationary value when the system attains a global order.

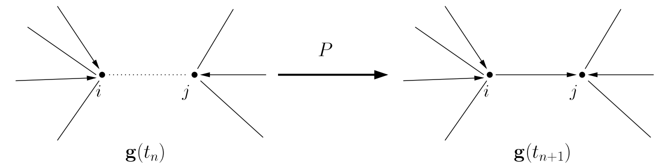

Fig. 1. Schematic diagram for evolution of the network: A portion of the network $\mathbf{g}(t)$ is shown. Arrows, pointing towards the nodes denote inward links and the lines denote the outward links for a randomly chosen pair of nodes (i, j). Since, $g_{i,j}(t_{n+1}) = 0$, a link is established at time $t = t_{n+1}$ with probability P defined by Eq. (2). The network remains static, i.e. $\mathbf{g}(t) = \mathbf{g}(t_{n+1})$, in the interval $[t_{n+1}, t_{n+2})$. The converse can also happen provided there is already a link between the pair of nodes at t_{n+1}. For a linearly coupled system we have probability P given by Eq. (4).

3. Coupled oscillator systems

We apply the above model to a system of nonlinear oscillators. The transition probability for the evolution of the interaction topology can be written in terms of the oscillator and network degrees of freedom. Such a cumulative effect of topology and dynamics leads to the steady state in a self-organizing manner. We consider a collection of limit-cycle oscillators as our first example. We also consider a system of van der Pol oscillators which reduce to the former in the limit of small nonlinearity. The two oscillators we consider in our study have stable limit-cycle solutions in the uncoupled case. Below we formalize the notion of a system of linearly coupled oscillators each of them having stable limit cycle solutions.

3.1. Linearly coupled system

Consider a linearly coupled system where the dynamics is described by

$$\dot{\mathbf{x}} = \mathbf{f}_0(\mathbf{x}) + \epsilon\mathbf{L}(\mathbf{g})\mathbf{x}, \qquad \mathbf{x}(t_0) = \mathbf{x}_0, \tag{3}$$

for all $t \in [t_n, t_{n+1})$, $n = 0, 1, \ldots$, where the matrix $\mathbf{L}(\mathbf{g})$ describes coupling between the individual units and the parameter ϵ controls the strength of the coupling. For the case of a symmetric graph $\mathbf{L} = (\mathbf{I} - \mathbf{K}^{-1}\mathbf{g})$ with $\mathbf{K} := \mathrm{diag}(k_1, \ldots, k_N)$ where $k_i = \sum_j g_{ij}$, is the graph Laplacian. Let us assume that the individual units are identical and have similar behavior in the decoupled limit $\epsilon = 0$. We shall study the synchronization of linearly coupled system with and investigate how synchronization occurs as the coupling strength is varied.

The transition probabilities $P(\mathbf{g}'|\mathbf{g}(t_n); \mathbf{x}(t_{n+1}))$ which govern the evolution of the interaction topology $\mathbf{g}$ at $t = t_{n+1}$ are assumed to be separable in the dynamical and network degrees of freedom. Hence, $P(\mathbf{g}'|\mathbf{g}(t_n); \mathbf{x}(t_{n+1})) = \eta(\mathbf{g}'|\mathbf{g}(t_n))\rho(\mathbf{x}(t_{n+1}))$ for some functions η and ρ. As the oscillators involved in our study have two degrees of freedom, the radial and angular coordinates, we assume that the function ρ is further separable in the radial and angular degrees of freedom, i.e., $\rho(\mathbf{x}(t_{n+1})) = \rho_r(\mathbf{r}(t_{n+1}))\rho_\phi(\phi(t_{n+1}))$, where ρ_r and ρ_ϕ are the radial and angular probability densities and $\mathbf{r}$ and ϕ are the radial and angular coordinates of the collection of oscillators. As the interaction network is assumed to change by only one link at t_{n+1}, the transition probability can be written as:

$$P = (1 - g_{ij})\, \eta\, \rho_r\, \rho_\phi + g_{ij}(1 - \eta)(1 - \rho_r)(1 - \rho_\phi), \tag{4}$$

where the first term in Eq. (4) corresponds to an addition of link between nodes i and j and the second term corresponds to deletion(if previously attached) of a link. The transition is depicted schematically in Fig. 1.

The exact forms of the functions are defined in accordance with the degrees and the phase space coordinates associated with nodes i and j, i.e., $\eta = \eta(k_i^{\pm}, k_j^{\pm})$, $\rho_r = \rho_r(r_i, r_j)$ and $\rho_\phi = \rho_\phi(\phi_i, \phi_j)$. For