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Averaging theory at any order for computing limit cycles of discontinuous piecewise differential systems with many zones

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HIGHLIGHTS

- We provide the averaging theory at any order for a class of discontinuous systems.
- The main theorem allows to study the limit cycles of these systems.
- The main result is applied to study nonsmooth perturbations of nonlinear centers.
- For these centers we estimate the number of limit cycles.

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ABSTRACT

This work is devoted to study the existence of periodic solutions for a class of ε-family of discontinuous differential systems with many zones. We show that the averaged functions at any order control the existence of crossing limit cycles for systems in this class. We also provide some examples dealing with nonsmooth perturbations of nonlinear centers.

1. Introduction and statement of the main results

In the qualitative theory of real planar differential system the determination of limit cycles, defined by Poincaré [1], has become one of the main problems. The second part of the 16th Hilbert problem deals with planar polynomial vector fields and proposes to find a uniform upper bound $H(n)$ (called Hilbert's number) for the number of limit cycles that these vector fields can have depending only on the polynomial degree n. The averaging method has been used to provide lower bounds for the Hilbert number $H(n)$ see, for instance, [2]. The interest on this topic extends to what we call *discontinuous piecewise smooth vector fields*.

The increasing interest in the theory of nonsmooth vector fields has been mainly motivated by its strong relation with Physics, Engineering, Biology, Economy, and other branches of science. In fact, discontinuous piecewise smooth differential systems are very useful to model phenomena presenting abrupt switches such as electronic relays, mechanical impact, and neuronal networks, see for instance [3–5]. The extension of the averaging theory to discontinuous piecewise smooth vector field has been the central subject of investigation of the following works [6–9].

A piecewise smooth vector field defined on an open bounded set $U \subset \mathbb{R}^n$ is a function $F : U \to \mathbb{R}^n$ which is continuous except on a set Σ of measure 0, called the *set of discontinuity* of the vector field F. It is assumed that $U \setminus \Sigma$ is a finite collection of disjoint open sets U_i, $i = 1, 2, \dots, m$, such that the restriction $F_i = F\big|_{U_i}$ is continuously extendable to the compact set $\overline{U_i}$. The local trajectory of F at a point $p \in U_i$ is given by the usual notion. However the local trajectory of F at a point $p \in \Sigma$ needs to be given with some care. In [10], taking advantage of the theory of differential inclusion (see [11]), Filippov established some conventions for what would be a local trajectory at points of discontinuity where the set Σ is locally a codimension one embedded submanifold of $\mathbb{R}^n$. For a such point $p \in \Sigma$, we consider a sufficiently small neighborhood U_p of p such that Σ splits $U_p \setminus \Sigma$ in two disjoint open sets U_p^+ and U_p^- and

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denote $F^{\pm}(p) = F\big|_{U_p^{\pm}}(p)$. In short, if the vectors $F^{\pm}(p)$ point at the same direction then the local trajectory of F at p is given as the concatenation of the local trajectories of $F^{\pm}$ at p. In this case we say that the trajectory *crosses* the set of discontinuity and that p is a *crossing point*. If the vectors $F^{\pm}(p)$ point in opposite directions then the local trajectory of F at p slides on Σ. In this case we say that p is a *sliding point*. For more details on the Filippov conventions see [10,12].

In this paper we are interested in establishing conditions for the existence of crossing limit cycles for a class of discontinuous piecewise smooth vector fields, that is limit cycles which only cross the set of discontinuity Σ. It is worth to say that if Σ is locally described as $h^{-1}(0)$, being $h : U \to \mathbb{R}$ a smooth function and 0 a regular value, then $\langle \nabla h(p), F^{+}(p)\rangle\langle \nabla h(p), F^{-}(p)\rangle > 0$ is the condition in order that p is a crossing point. For nonautonomous system the same definition can be applied considering the *extended phase space* where the system becomes autonomous by taking the time as a new space variable with constant velocity equal to 1.

In the sequel we introduce a short review of the averaging theory for computing isolated periodic solutions of differential equations. Then we set the class of nonautonomous discontinuous piecewise smooth differential equations that we are interested as well as our main result (Theorem 1). After that the rest of this section is devoted to present a class of planar autonomous discontinuous piecewise smooth differential systems that can be studied using our main result. We stress that this last class of systems stands as a motivation for this work.

1.1. Background on the averaging theory for smooth systems

Let D be an open bounded subset of $\mathbb{R}_+$ and denote $\mathbb{S}^1 \equiv \mathbb{R}/(2\pi\mathbb{Z})$. Consider C^{k+1} functions $F_i : \mathbb{S}^1 \times D \to \mathbb{R}$ for $i = 0, 1, 2, \dots, k$, and $R : \mathbb{S}^1 \times D \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}$. Note that $\theta \in \mathbb{S}^1 \equiv \mathbb{R}/(2\pi\mathbb{Z})$ means that the above functions are 2π-periodic in the variable θ. Now consider the following differential equation

$$r'(\theta) = \sum_{i=0}^{k} \varepsilon^i F_i(\theta, r) + \varepsilon^{k+1} R(\theta, r, \varepsilon), \tag{1}$$

and assume that the solution $\varphi(\theta, \rho)$ of the *unperturbed system* $r'(\theta) = F_0(\theta, r)$, such that $\varphi(0, \rho) = \rho$, is 2π-periodic for every $\rho \in D$. Here the prime denotes the derivative in the variable θ.

A central question in the study of system (1) is to understand which periodic orbits of the unperturbed system $r'(\theta) = F_0(\theta, r)$ persist for $|\varepsilon| \neq 0$ sufficiently small. In others words to provide sufficient conditions for the persistence of isolated periodic solutions. The averaging theory is one of the best tools to track this problem. Summarizing, it consists in defining a collection of functions $f_i : D \to \mathbb{R}$, for $i = 1, 2, \dots, k$, called *averaged functions*, such that their simple zeros provide the existence of isolated periodic solutions of the differential equation (1). In [13,14] it was proved that these averaged functions are

$$f_i(\rho) = \frac{y_i(2\pi, \rho)}{i!}, \tag{2}$$

where $y_i : \mathbb{R} \times D \to \mathbb{R}$ for $i = 1, 2, \dots, k$, are defined recurrently by the following integral equations

$$\begin{aligned} y_1(\theta, \rho) &= \int_0^{\theta} \Big(F_1(\phi, \varphi(\phi, \rho)) + \partial F_0(\phi, \varphi(\phi, \rho)) y_1(\phi, \rho)\Big) d\phi, \\ y_i(\theta, \rho) & \\ &= i! \int_0^{\theta} \left(F_i(\phi, \varphi(\phi, \rho)) + \sum_{l=1}^{i} \sum_{S_l} \frac{1}{b_1!\, b_2! 2!^{b_2} \cdots b_l! l!^{b_l}} \right. \\ &\quad \left. \cdot\, \partial^L F_{i-l}(\phi, \varphi(\phi, \rho)) \prod_{j=1}^{l} y_j(\phi, \rho)^{b_j} \right) d\phi, \ \text{ for } i = 2, \dots, k. \end{aligned} \tag{3}$$

Here $\partial^L G(\phi, \rho)$ denotes the derivative order L of a function G with respect to the variable ρ, and S_l is the set of all l-tuples of non-negative integers $(b_1, b_2, \dots, b_l)$ satisfying $b_1 + 2b_2 + \dots + lb_l = l$, and $L = b_1 + b_2 + \dots + b_l$.

When one considers the above problem in the world of discontinuous piecewise smooth differential systems it is not always true that the higher averaged functions (2) allow to study the persistence of isolated periodic solutions. In [7,9] this problem was considered for general Filippov systems when $F_0(\theta, r) \equiv 0$ and it was proved that the averaged function of first order can provide information on the existence of crossing isolated periodic solutions. Furthermore the authors have found conditions on those systems in order to assure that the averaged function of second order also provides information on the existence of crossing isolated periodic solutions. When $F_0(\theta, r) \not\equiv 0$ but the solutions of the unperturbed system $\dot r = F_0(\theta, r)$ are 2π-periodic the authors in [8] have found conditions on those systems in order to assure that the averaged function of first order provides information on the existence of crossing isolated periodic solutions.

1.2. Standard form and main result

In what follows we introduce a class of discontinuous nonautonomous piecewise smooth differential equations for which the averaged functions (2) at any order provide information on the existence of isolated periodic solutions.

Let $n > 1$ be a positive integer, $\alpha_0 = 0$, $\alpha_n = 2\pi$ and $\alpha = (\alpha_1, \dots, \alpha_{n-1}) \in \mathbb{T}^{n-1}$ a $(n-1)$-tuple of angles such that $0 = \alpha_0 < \alpha_1 < \alpha_2 < \dots < \alpha_{n-1} < \alpha_n = 2\pi$. For $i = 0, 1, \dots, k$ and $j = 1, 2, \dots, n$, let $F_i^j : \mathbb{S}^1 \times D \to \mathbb{R}$ and $R^j : \mathbb{S}^1 \times D \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}$ be C^{k+1} functions, where D is an open bounded interval of $\mathbb{R}_+$ and $\mathbb{S}^1 \equiv \mathbb{R}/(2\pi\mathbb{Z})$. Denote

$$\begin{aligned} F_i(\theta, r) &= \sum_{j=1}^{n} \chi_{[\alpha_{j-1}, \alpha_j]}(\theta) F_i^j(\theta, r), \ \ i = 0, 1, \dots, k, \quad \text{and} \\ R(\theta, r, \varepsilon) &= \sum_{j=1}^{n} \chi_{[\alpha_{j-1}, \alpha_j]}(\theta) R^j(\theta, r, \varepsilon), \end{aligned} \tag{4}$$

where $\chi_A(\theta)$ denotes the characteristic function of an interval A:

$$\chi_A(\theta) = \begin{cases} 1 & \text{if } \theta \in A, \\ 0 & \text{if } \theta \notin A. \end{cases}$$

We note that $\theta \in \mathbb{S}^1 \equiv \mathbb{R}/(2\pi\mathbb{Z})$ means that the above functions are 2π-periodic in the variable θ.

This work is devoted to study the existence of isolated periodic solutions of the following discontinuous nonautonomous 2π-periodic piecewise smooth differential equation

$$r'(\theta) = \sum_{i=0}^{k} \varepsilon^i F_i(\theta, r) + \varepsilon^{k+1} R(\theta, r, \varepsilon). \tag{5}$$

In this case the set of discontinuity is given by $\Sigma = (\{\theta = 0 \equiv 2\pi\} \cup \{\theta = \alpha_1\} \cup \dots \cup \{\theta = \alpha_{n-1}\}) \cap \mathbb{S}^1 \times D$. In short, we shall provide sufficient conditions in order to show that, for $|\varepsilon| \neq 0$ sufficiently small, the averaged functions (2) at any order can be used to ensure the existence of crossing limit cycles. It is worth to mention that the Lth derivative of the discontinuous function F_i with respect to the second variable, $\partial^L F_i(\theta, r)$, which appears in the averaged functions (2), is given by

$$\partial^L F_i(\theta, r) = \sum_{j=1}^{n} \chi_{[\alpha_{j-1}, \alpha_j]}(\theta) \partial^L F_i^j(\theta, r), \ \ i = 0, 1, \dots, k.$$

Denote by $\varphi(\theta, \rho)$ the solution of the system $r'(\theta) = F_0(\theta, r)$ such that $\varphi(0, \rho) = \rho$. From now on this last system will be called *unperturbed system*. We assume the following hypothesis:

(H1) For each $\rho \in D$ the solution $\varphi(\theta, \rho)$ is defined for every $\theta \in \mathbb{S}^1$, it reaches Σ only at crossing points, and it is 2π-periodic.