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Multivariate Hadamard self-similarity: Testing fractal connectivity

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HIGHLIGHTS

- A new multivariate Gaussian scale-free and fractal connectivity stochastic model.
- Asymptotic performance study of multivariate DWT estimators of scaling exponents.
- Approximate confidence interval construction for scaling exponent estimators.
- Statistical test for the presence of fractal connectivity from a single sample path.

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ABSTRACT

While scale invariance is commonly observed in each component of real world multivariate signals, it is also often the case that the inter-component correlation structure is not fractally connected, i.e., its scaling behavior is not determined by that of the individual components. To model this situation in a versatile manner, we introduce a class of multivariate Gaussian stochastic processes called Hadamard fractional Brownian motion (HfBm). Its theoretical study sheds light on the issues raised by the joint requirement of entry-wise scaling and departures from fractal connectivity. An asymptotically normal wavelet-based estimator for its scaling parameter, called the Hurst matrix, is proposed, as well as asymptotically valid confidence intervals. The latter are accompanied by original finite sample procedures for computing confidence intervals and testing fractal connectivity from one single and finite size observation. Monte Carlo simulation studies are used to assess the estimation performance as a function of the (finite) sample size, and to quantify the impact of omitting wavelet cross-correlation terms. The simulation studies are shown to validate the use of approximate confidence intervals, together with the significance level and power of the fractal connectivity test. The test performance and properties are further studied as functions of the HfBm parameters.

1. Introduction

1.1. Scale invariance

The relevance of the paradigm of scale invariance is evidenced by its successful use, over the last few decades, in the analysis of the dynamics in data obtained from a rather diverse spectrum of real world applications. The latter range from natural phenomena – physics (hydrodynamic turbulence [1], out-of-equilibrium physics), geophysics (rainfalls), biology (body rhythms [2], heart rate [3,4], neurosciences and genomics [5–8]) – to human activity – Internet traffic [9,10], finance [11], urban growth and art investigation [12–14].

In essence, scale invariance – also called scaling, or scale-free dynamics – implies that the phenomenical or phenomenological dynamics are driven by a large continuum of equally important time scales, rather than by a small number of characteristic scales. The investigation's focus is on identifying a relation amongst relevant scales rather than picking out characteristic scales.

Historically, self-similarity was one of the first proposed mathematical frameworks for the modeling of scale invariance (e.g., [15]). A random system is called self-similar when dilated copies of a single signal X are statistically indistinguishable, namely,

$$\{X(t)\}_{t\in\mathbb{R}} \overset{\text{fdd}}{=} \{a^H X(t/a)\}_{t\in\mathbb{R}}, \quad \forall a > 0, \tag{1.1}$$

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where $\stackrel{\text{fdd}}{=}$ stands for the equality of finite-dimensional distributions. An example of a stochastic process that satisfies the property (1.1) is fractional Brownian motion (fBm). Indeed, the latter is the only self-similar, Gaussian, stationary increment process, and it is the most widely used scaling model for real-world signals [16].

Starting from (1.1), the key parameter for quantifying scale-free dynamics is the scaling, or Hurst, exponent $0 < H < 1$. The estimation of H is the central task in scaling analysis, and it has received considerable effort and attention in the last three decades (see [17] for a review). The present contribution is about wavelet-based estimation [18,19]. It relies on the key scaling property

$$\frac{1}{T}\sum_t T_X^2(a,t) \simeq Ca^{\alpha}, \quad \alpha := 2H, \tag{1.2}$$

where $T_X(a,t)$ is the wavelet coefficient of an underlying self-similar stochastic process and T is the number of available coefficients. In other words, the sample wavelet variance of the stochastic process behaves like a power law with respect to the scale a.

1.2. Multivariate scaling

In many modern fields of application such as Internet traffic and neurology, data is collected in the form of multivariate time series. Univariate-like analysis in the spirit of (1.2) – i.e., independently on each component – does not account for the information stemming from correlations across components. The classical fBm parametric family, for example, provides at best a model for component-wise scaling, and thus cannot be used as the foundation for a multivariate modeling paradigm.

To model self-similarity in a multivariate setting, a natural extension of fBm, called Operator fractional Brownian motion (OfBm), was recently defined and studied (see [20–22]). An OfBm $\underline{X}$ satisfies the m-variate self-similarity relation

$$\{\underline{X}(t)\}_{t\in\mathbb{R}} \stackrel{\text{fdd}}{=} \{a^{\underline{\underline{H}}}\underline{X}(t/a)\}_{t\in\mathbb{R}}, \quad \forall a > 0, \tag{1.3}$$

where the scaling exponent is a $m \times m$ matrix $\underline{\underline{H}}$, and $a^{\underline{\underline{H}}}$ stands for the matrix exponential $\sum_{k=0}^{\infty}(H\log a)^k/k!$. Likewise, the wavelet spectrum of each individual component is not a single power law as in (1.2); instead, it behaves like a mixture of distinct univariate power laws. In its most general form, OfBm remains scarcely used in applications; recent efforts have tackled many difficulties that arise in the identification of its parameters [23,24].

1.3. Entry-wise multivariate scaling

We call an OfBm entry-wise scaling when the Hurst parameter is simply a diagonal matrix $\underline{\underline{H}} = \text{diag}(H_1,\ldots,H_m)$. This instance of OfBm has been used in many applications (e.g., [6,25]) and its estimation is thoroughly studied in [20]. Since $\underline{\underline{H}}$ is diagonal, the relation (1.3) takes the form

$$\{X_1(t),\ldots,X_m(t)\}_{t\in\mathbb{R}} \stackrel{\text{fdd}}{=} \{a^{H_1}X_1(t/a),\ldots,a^{H_m}X_m(t/a)\}_{t\in\mathbb{R}}, \quad \forall a > 0, \tag{1.4}$$

which is reminiscent of the univariate case. This implies that the extension of (1.2) to all auto- and cross-components of m-variate data can be written as

$$\frac{1}{T}\sum_t T_{X_{q_1}}(a,t)T_{X_{q_2}}(a,t) \simeq Ca^{\alpha_{q_1q_2}}, \quad \alpha_{q_1q_2} := H_{q_1} + H_{q_2}, \quad q_1, q_2 = 1,\ldots,m, \tag{1.5}$$

where T is as in (1.2).

1.4. Fractal connectivity

Yet, entry-wise scaling OfBm is a restrictive model since the cross-scaling exponents $\alpha_{q_1q_2}$, $q_1 \neq q_2$, are determined by the auto-scaling exponents $\alpha_{q_1q_1}$ and $\alpha_{q_2q_2}$, i.e.,

$$\alpha_{q_1q_2} = H_{q_1} + H_{q_2} = (\alpha_{q_1q_1} + \alpha_{q_2q_2})/2. \tag{1.6}$$

In this situation, called *fractal connectivity* [6,25,26], no additional scaling information can be extracted from the analysis of *cross-components*. However, in real world applications, cross-components are expected to contain information on the dynamics underlying the data, e.g., cross-correlation functions. As an example, recent investigation of multivariate brain dynamics in [6] produced evidence of departures from fractal connectivity, notably for subjects carrying out prescribed tasks. Other fields where cross-correlation or non-fractally connected modeling has been pursued include physics [27–36] and econometrics [37–39]. This means that there is a clear need for more versatile models than entry-wise scaling OfBm (see also Remark 2.2). The covariance structure of the new model should satisfy the following two requirements:

1. all auto- and cross-components are (approximately) self-similar;
2. departures from fractal connectivity are allowed, i.e., the exponents of the cross-components are not necessarily determined by the exponents of the corresponding auto-components.

Hereinafter, a departure from fractal connectivity (1.6) on a given covariance structure entry (q_1, q_2) will be quantified by means of the parameter

$$\delta_{q_1q_2} = \frac{\alpha_{q_1q_1} + \alpha_{q_2q_2}}{2} - \alpha_{q_1q_2} \geq 0, \quad q_1, q_2 = 1,\ldots,m, \tag{1.7}$$

where nonnegativeness is a consequence of the Cauchy–Schwarz inequality (see (2.10)). It is clear that $\delta_{q_1q_2} = 0$ when $q_1 = q_2$.