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Time-dependent spectral renormalization method

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Abstract

The spectral renormalization method was introduced by Ablowitz and Musslimani in 2005, [Opt. Lett. **30**, pp. 2140-2142] as an effective way to numerically compute (time-independent) bound states for certain nonlinear boundary value problems. In this paper, we extend those ideas to the *time* domain and introduce a *time-dependent spectral renormalization* method as a numerical means to simulate linear and nonlinear evolution equations. The essence of the method is to convert the underlying evolution equation from its partial or ordinary differential form (using Duhamel's principle) into an integral equation. The solution sought is then viewed as a fixed point in both space and time. The resulting integral equation is then numerically solved using a simple renormalized fixed-point iteration method. Convergence is achieved by introducing a *time-dependent* renormalization factor which is numerically computed from the *physical properties* of the governing evolution equation. The proposed method has the ability to incorporate physics into the simulations in the form of conservation laws or dissipation rates. This novel scheme is implemented on benchmark evolution equations: the classical nonlinear Schrödinger (NLS), integrable PT symmetric nonlocal NLS and the viscous Burgers' equations, each of which being a prototypical example of a conservative and dissipative dynamical system. Numerical implementation and algorithm performance are also discussed.

1. Introduction

Computational methods play an indispensable role in various branches of the physical [1], chemical and biological [2, 3] sciences. In many cases the underlying phenomenon being studied is modeled by either a single or a set of ordinary or partial differential equations. Well-known examples include the quantum Schrödinger equation governing the time evolution of a quantum wave function [4], reaction-diffusion type systems [5] that describe chemical reaction or population dynamics (such as swimming microorganism), and the Navier-Stokes equation modeling the motion of an incompressible Newtonian fluid [6] to mention a few.

Over the years, numerical tools have played an ever increasing role in the advancement of scientific discoveries. They provide a unique opportunity to tackle many challenging scientific problems that are otherwise difficult to solve. This is the case, for instance, in complex turbulent flows, weather prediction, stochastic neural dynamics and many body physics with large degrees of freedom. Surprisingly enough, computational methods and numerical simulations can actually ignite new fundamental ideas and ultimately lead to the development of new theories. One example that stands out is the concept of a soliton which emerged as a result of numerical experiments. Solitons, shape-invariant nonlinear waves that exhibit particle-like behavior upon collision, were discovered in 1965 by Zabusky and Kruskal [7] while performing numerical simulations on the Korteweg-de Vries equation [8]. Their findings sparked intense research interest in many areas of physics and mathematics which subsequently led to the establishment of the inverse scattering transform and integrable nonlinear evolution equations [9, 10]. Soon thereafter, the notion of soliton or solitary wave, spread to many diverse areas in the physical, chemical and biological sciences. Examples include, optical spatial and temporal solitons [11, 12], atomic Bose-Einstein condensates [13, 14], atomic chains [15], molecular and biophysical systems [16] and electrical lattices [17].

A unifying theme among such diverse fields is the development of computational methods capable of accurately and efficiently capturing the physics under study. As such, numerous numerical tools have been developed in the last few decades to simulate evolution equations of the nonlinear Schrödinger and Gross-Pitaevskii type [18, 19, 20, 21, 22, 23, 24]. Perhaps the simplest schemes are finite difference methods such as the Crank-Nicolson algorithm [20, 21, 22]. Among the easiest to implement are the well-known Runge-Kutta integrators [23]. For evolution equations that can be written as a coupled system of linear and nonlinear equations, both of which can be solved exactly, time-splitting methods are fast and typically quite accurate [18, 24]. For stiff problems the so-called exponential differencing methods [25, 26] are effective. Here, the underlying evolution equation is written in an integral form which is then solved using exponential time-stepping schemes [27, 28, 29]. An issue of great importance is how to devise numerical schemes capable of enforcing physics into the simulations. For example, when