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# Global existence of solutions to a tear film model with locally elevated evaporation rates

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**Abstract**

Motivated by a model proposed by Peng et al. [*Advances in Coll. and Interf. Sci.* 206 (2014)] for break-up of tear films on human eyes, we study the dynamics of a generalized thin film model. The governing equations form a fourth-order coupled system of nonlinear parabolic PDEs for the film thickness and salt concentration subject to non-conservative effects representing evaporation. We analytically prove the global existence of solutions to this model with mobility exponents in several different ranges and present numerical simulations that are in agreement with the analytic results. We also numerically capture other interesting dynamics of the model, including finite-time rupture-shock phenomenon due to the instabilities caused by locally elevated evaporation rates, convergence to equilibrium and infinite-time thinning.

*Keywords:* global existence, tear film, fourth-order nonlinear partial differential equations, rupture, thin film equation, evaporation, osmolarity, finite-time singularity

## 1. Introduction

In this article, we study the regularity of solutions to a one-dimensional nonlinear partial differential equation system for a fluid film height $h(x,t)$ and salt concentration (also called the osmolarity) $s(x,t)$ on a finite domain, $0 \le x \le L$,

$$h_t = -(h^n h_{xxx})_x - h^m(\bar{S} - s), \tag{1.1a}$$

$$s_t = s_{xx} + \left(\frac{h_x}{h} - h^{n-1}h_{xxx}\right) s_x + s(\bar{S} - s)h^{m-1}. \tag{1.1b}$$

This family of PDEs is motivated by a non-conservative lubrication model for evaporating tear films on human eyes. Based on the model proposed by Peng et al. [22], a spatial variation in a thin lipid layer on the tear film leads to locally elevated evaporation rates of the tear film, which in turn affects the local salt concentration in the liquid film. In our model (1.1) the influences of the lipid layer thickness on osmolarity are included in the effective salt capacity function, $\bar{S}(x) \in L^\infty([0, L])$. This will be taken to be a given positive function with increased values over some portion of the domain, corresponding to elevated evaporation rates (and decreased lipid concentrations). Starting from initial data $(h_0(x), s_0(x))$ at time $t = 0$ which satisfy $h_0 > 0$ and $0 < s_0 \le \|\bar{S}\|_\infty$, the dynamics will be subject to no-flux and normal-contact boundary conditions

$$h_x(0) = h_x(L) = 0, \quad h_{xxx}(0) = h_{xxx}(L) = 0, \quad s_x(0) = s_x(L) = 0. \tag{1.1c}$$

The mobility exponents $n$ and $m$ in (1.1) are introduced to analyze and separate the influences of the conservative and non-conservative fluxes in the model respectively. The term $(h^n h_{xxx})_x$ in the model is due to capillary forces, the terms $h^m(\bar{S} - s)$ and $s(\bar{S} - s)h^{m-1}$ are related to evaporative effects, $s_{xx}$ corresponds to the diffusion of the

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