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Overhanging of membranes and filaments adhering to periodic graph substrates

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HIGHLIGHTS

- One-dimensional models of filaments and membranes adhering to periodic substrates are studied.
- The effects of bending and adhesion are taken into account.
- Whether global minimizers are graphs or overhanging is mainly considered.
- Ranges of characteristic parameters ensuring the presence and absence of overhangs are obtained.

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ABSTRACT

This paper mathematically studies membranes and filaments adhering to periodic patterned substrates in a one-dimensional model. The problem is formulated by the minimizing problem of an elastic energy with a contact potential on graph substrates. Global minimizers (ground states) are mainly considered in view of their graph representations. Our main results exhibit sufficient conditions for the graph representation and examples of situations where any global minimizer must overhang.

1. Introduction

The figuration of elastic bodies is complicated to comprehend, in particular, if external factors and constraints are taken into consideration. This paper is devoted to a theoretical study of slender elastic bodies adhering to solid substrates.

The contact and adhesion problems between soft objects and solid substrates appear in various contexts. For example, complex adhesion patterns are observed when soft nano-objects, as graphene [1,2] or carbon nanotubes [3], are sheeted on rough patterned substrates. The adhesion property is also known for vesicles (cf. [4]). More broadly, in contact mechanics [5], it is a central question to ask how elastic bodies contact rough substrates [6,7]. This question is relevant for many motivating problems as rubber friction [8] or adhesion in biological systems as geckos [9–11]. Recently, there are remarkable progresses in "elasto-capillary" problems [12]. The elasto-capillary problems essentially relate to our problem in the sense that they are focused on the competition between elasticity and adhesiveness.

1.1. Our model

This paper mathematically studies the adhesion problems of filaments and membranes in a one-dimensional setting, as in [13]. To be more precise, we consider the minimizing problem of the energy

$$\mathcal{E}[\gamma] = \int_\gamma ds \left[\frac{C}{2}\kappa^2(s) + \sigma\left(\gamma(s)\right) \right] \tag{1.1}$$

defined for planar curves γ. Here κ and s denote the curvature and arc length parameter, respectively. Admissible curves γ (corresponding to elastic bodies) are constrained in the upper side of a given λ-periodic substrate function ψ_λ as in Fig. 1. The constant $C > 0$ corresponds to the bending rigidity. The contact potential σ is defined as $\sigma = \sigma_F$ in the free part and $\sigma = \sigma_B$ in the bounded part, where $0 < \sigma_B < \sigma_F$ are constants. The constants σ_B and σ_F correspond to tension or surface energies. (See Section 2 for details.)

Our energy is a simple generalization of the modified total squared curvature, so-called Euler's elastica energy (see [14–18]

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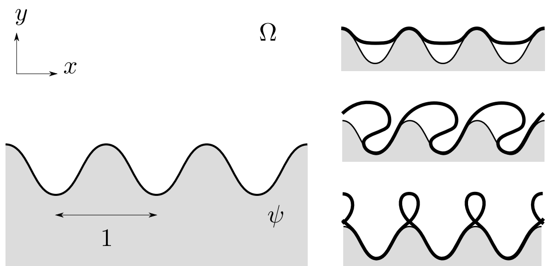

Fig. 1. Periodic substrate function ψ and periodic admissible curves. Admissible curves may overhang or self-intersect.

and also [19–21]), so that an adhesion effect (contact potential) is included. Its minimization invokes a free boundary problem of the elastica equation, i.e., the free part of any minimizing curve satisfies the curvature equation $C(\kappa_{ss} + \kappa^3/2) - \sigma_F\kappa = 0$. The free boundary conditions are concerned with curvature jumps (see [13] and also [4,12,22,23]). Our model can be regarded as an elastic version of wetting problems (cf. [24,25]).

Our model concerns only the bending modes of filaments or membranes and neglects the stretching modes. As mentioned in [13], the underlying physical assumptions are that elastic bodies are sufficiently thin, vary only in one direction, and move along substrates freely (no friction). The stretching modes should be taken into account in fully two-dimensional models, even for thin films without friction (see e.g. [2,26]).

1.2. Our goal

The local laws (as the elastica equation or boundary conditions) are well-known in our model since similar models have been widely studied (e.g. in [4,12,22,23]). The fundamental goal of this paper is to know the whole shapes of minimizers in our model. However, it is not realistic to determine the exact whole shapes of minimizers for arbitrary parameters and a substrate. This paper focuses on whether minimizers are represented by the graphs of functions or not.

Whether minimizers are graphs or have overhangs is an important consequence on the shapes. In fact, the absence of overhangs guarantees that the shape of a solution is not so "complex", in particular, there is no self-intersection. Conversely, the presence of overhangs implies the possibility of self-intersections. If once membranes or filaments self-intersect, then other mechanisms (not taken into account in our model) may yield more complex shapes as rackets [27–29] (see also [12]).

An a priori guarantee of the graph representation is also important for the theoretical study. Such a guarantee rigorously justifies the graph setting, i.e., the assumption to consider only graph curves as admissible curves. The graph setting yields strong topological and morphological constraints, and hence makes the analysis considerably simpler. In fact, there are theoretical studies [13,30,31] concerning the whole shapes of minimizers in our model, but all of them rely on the graph setting. The paper [13] particularly depends on the graph setting since its analysis crucially uses the small slope approximation.

1.3. Main results

The present paper gives the first rigorous study on the graph representations of global minimizers (ground states). A theoretical reason to consider only global minimizers is that the shapes of local minimizers (metastable states) may be more complicated even for parameters ensuring the graph representations of global minimizers (see Section 5 for details). The assumption of global minimality would be however appropriate for some experimental situations, for example, thin films on substrates with wetting fluids at the interfaces (almost no friction) as in [26]. In addition, as a mathematical assumption, the present paper assumes that curves γ and a substrate ψ_λ have a same period λ.

To describe our results, it is convenient to recall the typical length scale $\ell = \sqrt{C/\sigma_F}$, which compares bending rigidity and surface tension. The scale ℓ is called the elasto-capillary length e.g. in [12,26]. As mentioned in [12,26], the scale ℓ appears as a typical bending scale of an elastic body. We also use the length scale $r = \|\psi_\lambda''\|_\infty^{-1}$ which is the reciprocal of the maximum of the second derivative. The scale r roughly corresponds to the minimal bending scale of ψ_λ. Moreover, the dimensionless ratio $\alpha = \sigma_B/\sigma_F$ is also important since it corresponds to adhesiveness.

Global minimizers are flat in many limiting cases; dominant bending effect ($C = \infty$), no adhesion ($\sigma_B = \sigma_F$) or flat substrate ($\psi = 0$). Hence, the graph representation is expected at least nearly the above cases. Indeed, Theorems 3.3 and 3.4 give explicit conditions ensuring that global minimizers must be graphs. The first condition is described as $1 - \alpha^{-1} \ll (\ell/\lambda)^2$. In particular, this condition is satisfied as the limits $C \to \infty$ and $\sigma_B \to \sigma_F$. The second condition is described as $(r/\lambda)^2 \gg \alpha^{-1} + (\ell/\lambda)^{-2}$. In particular, this condition is satisfied as the limit $r \to \infty$, which means a second order flatness of ψ_λ. Our proof uses only energy arguments; we compare the energies of all non-graph curves and special graph competitors.

On the other hand, even if ψ is smooth of class C^∞, it turns out that there are situations such that global minimizers are overhanging, i.e., not represented by graphs. The mechanism of overhangs is involved, so we deal with only special substrates like "fakir carpets" (see the figures in Section 4). Our result indicates that the wave height length scale H and dimensionless "deviation" $\Delta := \min\{\lambda, H\}/(\lambda + 2H)$ of a fakir carpet appear as characteristic quantities. More precisely, as a main result (Theorem 4.4), we rigorously prove that global minimizers must overhang if ψ_λ is smooth but shaped like a fakir carpet and moreover the relations $r \ll \ell \ll \min\{\lambda, H\}$ and $\alpha \ll \Delta$ are satisfied. Our proof is based on a geometric viewpoint to classify possible global states of non-overhanging curves, and an energy estimate for each of the cases. A special overhanging competitor is then constructed in view of the optimal bending scale ℓ. We notice that the condition $r \ll \ell$ requires that ℓ is not arbitrary small for overhangs. However, we also prove that if such substrates are Lipschitz (i.e., folding singularly $r = 0$), then ℓ can be arbitrary small for a fixed substrate (Theorem 4.7). To this end, we need further discussion for local bending structure, but we still use only energy arguments.

1.4. Related mathematical results

In the rest of this section, reviewing related mathematical literature, we see that in our one-dimensional problem both the contact potential and the total squared curvature play crucial roles for overhangs.

There is much mathematical literature of first order energies with contact potentials on flat substrates (see e.g. [32–35] for graphs, [25,33] for the boundary of sets, and references therein). The problems in the cited papers roughly correspond to our problem with $C = 0$ and $\psi_\lambda \equiv 0$ (but in higher dimensions). In first order cases, solutions may have edge singularities at the free boundary and the contact angle θ satisfies Young's equation $\cos\theta = \sigma_B/\sigma_F$. In higher dimensional cases, the contact potential may imply the loss of graph representation even in first order cases (cf. [34]). However, although our substrates are not flat, our problem is one-dimensional and periodic, so the graph setting would be still suitable while $C = 0$.

To our knowledge, there is little mathematical literature of higher order problems with contact potentials except the