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Abstract

This article presents the complete study of the long-time evolution of random waves of a vibrating thin elastic plate in the limit of small plate deformation so that modes of oscillations interact weakly. According to the wave turbulence theory a nonlinear wave system evolves in longtime creating a slow redistribution of the spectral energy from one mode to another. We derive step by step, following the method of cumulants expansion and multiscale asymptotic perturbations, the kinetic equation for the second order cumulants as well as the second and fourth order renormalization of the dispersion relation of the waves. We characterize the non-equilibrium evolution to an equilibrium wave spectrum, which happens to be the well known Rayleigh-Jeans distribution. Moreover we show the existence of an energy cascade, often called the Kolmogorov-Zakharov spectrum, which happens to be not simply a power law, but a logarithmic correction to the Rayleigh-Jeans distribution. We perform numerical simulations confirming these *scenarii*, namely the equilibrium relaxation for closed systems and the existence of an energy cascade wave spectrum. Both show a good agreement between theoretical predictions and numerics. We show also some other relevant features of vibrating elastic plates, such as the existence of a self-similar wave action inverse cascade which happens to blow-up in finite time. We discuss the mechanism of the wave breakdown phenomena in elastic plates as well as the limit of strong turbulence which arises as the thickness of the plate vanishes. Finally, we discuss the role of dissipation and the connection with experiments, and the generalization of the wave turbulence theory to elastic shells.

1. Introduction

Since more than fifty years it was established that long-time statistical properties of randomly fluctuating wavy systems possess a natural asymptotic closure because of the dispersive nature of the waves and the weakly nonlinear wave interaction [1, 2]. This so-called wave (or weak) turbulence theory (noted WTT later on) has proven to be a powerful method to study the evolution of nonlinear dispersive wave systems [3, 4, 5, 6]. As a main result, WTT predicts that the longtime dynamics is driven by a kinetic equation for the distribution of spectral densities. This method was first developed for surface gravity waves [1, 7], then for plasma waves [8], surface capillary waves [9] and nonlinear optics [10, 11, 12] among others.

The resulting kinetic equation has non-equilibrium properties similar to the usual Boltzmann equation for dilute gases, conserving the energy and the momentum. Moreover, it exhibits a H-theorem driving an isolated system towards equilibrium, characterized by the so-called Rayleigh-Jeans distribution. Most importantly, besides the elementary equilibrium (or thermodynamic) solution, Zakharov has shown [8] that non-equilibrium stationary solutions also arise which describe a constant flux transfer (or cascade) of a conserved quantities (*e.g.* energy) between large and small length scales. In the particular cases where these cascade solutions are power laws, they are named Kolmogorov–Zakharov (KZ) spectra.

Experimental evidences of KZ spectra have been found for ocean surface waves [13] and for capillary surfaces waves [14, 15, 16]. On the other hand, numerical simulations of surface waves have exhibited KZ spectrum for weak turbulent capillary waves [17] and, more recently, for gravity waves [18].

Ten years ago, we have shown using WTT that weak wave turbulence was also possible for elastic plates since they exhibit dispersive linear waves [19]. Indeed, adding inertia to the well known (static) theory of thin plates, one finds ballistic dispersive waves [20], which interact via cubic nonlinear terms that are weak if the plate deformations are small. Because of this cubic nonlinearity, the wave interaction mechanism involves four wave resonances and the kinetic equation has been deduced using the classical WTT framework. As a consequence, in addition to the Rayleigh-Jeans equilibrium distribution, the existence of spectra of direct energy cascade was proven theoretically and observed numerically.