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A numerical estimate of the regularity of a family of Strange Non-Chaotic Attractors

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HIGHLIGHTS

- We give an algorithm to compute regularities of SNA's based on tools of de la Llave–Petrov.
- It uses the Keller convergence construction to the attractor.
- It uses Daubechies Wavelets with 16 vanishing moments.
- The precision is two decimal digits compared with Weierstraß function.
- The loss of regularity as parameter changes is observed from wavelet coefficients.

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ABSTRACT

We estimate numerically the regularities of a family of Strange Non-Chaotic Attractors related with one of the models studied in (Grebogi et al., 1984) (see also Keller, 1996). To estimate these regularities we use wavelet analysis in the spirit of de la Llave and Petrov (2002) together with some ad-hoc techniques that we develop to overcome the theoretical difficulties that arise in the application of the method to the particular family that we consider. These difficulties are mainly due to the facts that we do not have an explicit formula for the attractor and it is discontinuous almost everywhere for some values of the parameters. Concretely we propose an algorithm based on the Fast Wavelet Transform. Also a quality check of the wavelet coefficients and regularity estimates is done.

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1. Introduction

The aim of this paper is to develop techniques and algorithms to compute approximations of (geometrically) extremely complicated dynamical invariant objects by means of wavelet expansions. Moreover, from the wavelet coefficients we want to derive an estimate of the regularity of these invariant objects. In the case when the theoretical regularity is known, the comparison between both values gives a natural and good quality test of the algorithms and approximations.

In this paper the invariant objects that we study and consider when developing our algorithms are Strange Non-chaotic Attractors. They appear in a natural way in families of quasiperiodically forced skew products on the cylinder of the form

$$\mathfrak{F}_{\sigma,\varepsilon}: \begin{array}{ccc} \mathbb{S}^1 \times \mathbb{R} & \longrightarrow & \mathbb{S}^1 \times \mathbb{R} \\ (\theta, x) & \longmapsto & (R_\omega(\theta), F_{\sigma,\varepsilon}(\theta, x)), \end{array} \qquad (1)$$

where $F_{\sigma,\varepsilon}: \mathbb{S}^1 \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous and $\mathcal{C}^1$ with respect to the second variable, $R_\omega(\theta) = \theta + \omega \pmod 1$ with $\omega \in \mathbb{R} \setminus \mathbb{Q}$, $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z} = [0, 1)$ denotes the circle and $\varepsilon, \sigma \in \mathbb{R}^+$. These systems have the important property that any fibre, $\{\theta\} \times \mathbb{R}$, is mapped into another fibre, $\{R_\omega(\theta)\} \times \mathbb{R}$.

Our main goal will be to derive approximations in terms of wavelets of the invariant maps $\varphi: \mathbb{S}^1 \longrightarrow \mathbb{R}$: $\varphi(R_\omega(\theta)) = F_{\sigma,\varepsilon}(\theta, \varphi(\theta))$. Under certain conditions the graphs of these invariant maps have very complicated geometry where roughly speaking, the word *complicated* means non-piecewise continuous. In such case, we will say that the graph of φ is a *Strange Non-chaotic Attractor (SNA)*. A usual particular case of SNA is when the invariant function is positive in a set of full Lebesgue measure and vanishes on a residual set.

A standard approach is to use Fourier expansions (rather than wavelet ones) when approximating dynamical invariant objects. In the SNA's framework this approach has a serious drawback: an accurate approximation of φ demands a high number of Fourier modes due to the appearance of strong oscillations (see e.g. [1]). One natural way to overcome this problem is by using other orthonormal bases such as wavelets and the multi-scale methods

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(see e.g. [2,3]). One of the advantages of this approach is that wavelets also define certain regularity spaces $\mathscr{B}^s_{\infty,\infty}$ (see e.g. [4,2,5,6]) that provide a natural framework for the approximations that one gets.

Precisely, the regularity can be considered as a trait of how φ becomes strange in terms of functional spaces. For example, in [7], the authors make numerical implementations of wavelet analysis to estimate the "*positive*" regularity of invariant objects which are graphs of functions in appropriate spaces. However, due to the complexity of the SNAs described above we need to consider the possibility that these objects have zero or even negative regularity (see [2]). Hence, the techniques of [7] need to be extended to this case. To this end, we develop ad-hoc techniques to overcome the theoretical difficulties of the objects we study in performing a wavelet analysis, in the same spirit of [7], to estimate the regularity of such φ. Our wavelet analysis will be based on the Fast Wavelet Transform (see e.g [3]).

The computation of the regularity (depending on parameters) can give some insight into the study of the fractalization or other routes of creation of SNA and help in detecting this bifurcation.

We apply the above program to a slight modification of the system considered in [8]. Indeed, the attractor obtained in [8] (as shown by Keller in [9]), is the graph of an upper semi-continuous function from the circle to $\mathbb{R}$ in the *pinched case* (that is, when there exists a fibre whose image is degenerate to a point), whereas in the non pinched one the attractor is the graph of a map with the same regularity as the skew product (see also [10,11]). As we will see, the wavelet coefficients together with the computed regularity detect well the functional space jump associated to the creation of the SNA.

This paper is organized in two parts. The first one is devoted to make a survey on wavelets and regularity. Whereas the second one deals with the application of these techniques to the SNA case. More concretely, in Section 2 we recall some topics about the theory of wavelet bases. In Section 3 we will review the notion of regularity through Besov functional spaces and discuss it by means of simple examples. In Besov spaces the regularity can be any real number (in contrast to Hölder regularity defined only for positive regularities). In Section 4 we survey the relation between the regularity and the wavelet coefficients of a function. Section 4.1 is devoted to present and test a methodology to numerically estimate regularities based on the previous sections.

Finally, in the second part, in Section 5 we present the family of Strange Non-Chaotic Attractors that we will study. In particular, we state Keller's Theorem and we emphasize some ideas on the proof. These ideas will be used in devising the algorithm that we propose. In Section 6, we present the techniques to overcome the theoretical difficulties arising from the SNA. In Section 6.3, we perform the algorithm to compute the regularity of the attractors and in Section 7, the results of this computation, for a particular instance of SNA's, are presented and discussed.

2. A survey on wavelets

We aim at approximating by means of wavelets a certain class of functions from the circle $\mathbb{R}/\mathbb{Z}$ to an interval of the real line. Recall that a standard approach used in the literature to compute and work with invariant objects of systems exhibiting periodic or quasi-periodic behaviour is to use finite Fourier approximations (trigonometric polynomials), namely functions of the form

$$\varphi(\theta) = a_0 + \sum_{n=1}^{N} (a_n \cos(n\theta) + b_n \sin(n\theta)) .$$

In this paper instead we aim at using finite wavelet expansions of the form:

$$\varphi(\theta) = a_0 + \sum_{j=0}^{J} \sum_{n=0}^{N_j} d_{j,n}\psi_{j,n}(\theta),$$

where $\psi_{j,n}(\theta)$ is obtained by translation and dilation of a mother wavelet $\psi(x)$. To be explicit, let us start by introducing the orthonormal wavelet basis of $\mathscr{L}^2(\mathbb{R})$. A natural way to do it is via the notion of Multiresolution Analysis. We refer the reader to [3,4] for more detailed and comprehensive expositions.

Definition 2.1. A sequence of closed subspaces $\{\mathcal{V}_j\}_{j\in\mathbb{Z}}$ of $\mathscr{L}^2(\mathbb{R})$ is a *Multiresolution Analysis* (or simply a *MRA*) if it satisfies the following six properties:

(a) $\{0\} \subset \cdots \subset \mathcal{V}_1 \subset \mathcal{V}_0 \subset \mathcal{V}_{-1} \subset \cdots \subset \mathscr{L}^2(\mathbb{R})$.
(b) $\{0\} = \bigcap_{j\in\mathbb{Z}} \mathcal{V}_j$.
(c) $\text{clos}\left(\bigcup_{j\in\mathbb{Z}} \mathcal{V}_j\right) = \mathscr{L}^2(\mathbb{R})$.
(d) There exists a function ϕ whose integer translates, $\phi(x-n)$, forms an orthonormal basis of $\mathcal{V}_0$. Such function is called the *scaling function*.
(e) For each $j \in \mathbb{Z}$ it follows that $f(x) \in \mathcal{V}_j$ if and only if $f(x-2^j n) \in \mathcal{V}_j$ for each $n \in \mathbb{Z}$.
(f) For each $j \in \mathbb{Z}$ it follows that $f(x) \in \mathcal{V}_j$ if and only if $f(x/2) \in \mathcal{V}_{j+1}$. ■

Before continuing the explanation, let us recall that for $f \in \mathscr{L}^2(\mathbb{R})$,

$$\widehat{f}(\xi) = \int_{\mathbb{R}} f(x)e^{-i\xi x}\, dx, \quad \xi \in \mathbb{R},$$

denotes the *Fourier transform* of f and $f^\vee(x)$

$$f^\vee(x) = \frac{1}{2\pi}\int_{\mathbb{R}} f(\xi)e^{i\xi x}\, d\xi, \quad x \in \mathbb{R}$$

stands for the *inverse Fourier transform*. If we fix an MRA, it follows that $\mathcal{V}_j$ has an orthonormal basis $\{\phi_{j,n}\}_{n\in\mathbb{Z}}$, for every j, where

$$\phi_{j,n}(x) = 2^{-j/2}\phi\left(\frac{x-2^j n}{2^j}\right).$$

Now, define the subspace $\mathcal{W}_j$ as the orthogonal complement of $\mathcal{V}_j$ on $\mathcal{V}_{j-1}$, that is,

$$\mathcal{V}_{j-1} = \mathcal{W}_j \oplus \mathcal{V}_j. \tag{2}$$

Therefore, by the inclusion of the spaces $\mathcal{V}_j$ we have

$$\mathscr{L}^2(\mathbb{R}) = \text{clos}\left(\bigoplus_{j\in\mathbb{Z}} \mathcal{W}_j\right) = \text{clos}\left(\mathcal{V}_0 \oplus \bigoplus_{j=-\infty}^{0} \mathcal{W}_j\right). \tag{3}$$

The *mother wavelet* $\psi \in \mathcal{W}_0$ is defined to be the function whose Fourier transform is

$$\widehat{\psi}(\xi) = \frac{1}{\sqrt{2}} e^{-i\xi}\widehat{h}^*(\xi+\pi)\widehat{\phi}(\xi) \tag{4}$$

where $\widehat{h}^*(\xi)$ is the complex conjugate of

$$\widehat{h}(\xi) = \sum_{n\in\mathbb{Z}} h[n]e^{-in\xi}, \tag{5}$$

with $\widehat{h}(0) = \sqrt{2}$ and $h[n] = \left\langle \frac{1}{\sqrt{2}}\phi\left(\frac{x}{2}\right), \phi(x-n)\right\rangle$ for $n \in \mathbb{Z}$. The sequence $h[n]$ is called the *scaling filter* (or the *low pass filter*) of the Multiresolution Analysis. We define the support of $h[n]$, denoted by supp(h), as the minimum subset $\mathfrak{I}$ of $\mathbb{Z}$ such that $\mathfrak{I} = \{\ell, \ell+1, \ldots, \ell'\}$ is a set of consecutive integers and

$$h[n] = 0 \quad \text{for every } n \in \mathbb{Z} \setminus \mathfrak{I}.$$

The following result (see [3, Theorem 7.3]) allows to obtain the wavelet basis from the scaling function:

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