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Characterizing complex networks through statistics of Möbius transformations

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Abstract

It is well-known now that dynamics of large populations of globally (all-to-all) coupled oscillators can be reduced to low-dimensional submanifolds (WS transformation and OA ansatz). Marvel et al. (*Chaos*, 2009.) described an intriguing algebraic structure standing behind this reduction: oscillators evolve by the action of the group of Möbius transformations.

Of course, dynamics in complex networks of coupled oscillators is highly complex and not reducible. Still, closer look unveils that even in complex networks some (possibly overlapping) groups of oscillators evolve by Möbius transformations. In this paper we study properties of the network by identifying Möbius transformations in the dynamics of oscillators. This enables us to introduce some new (statistical) concepts that characterize the network. In particular, the notion of *coherence* of the network (or subnetwork) is proposed.

This conceptual approach is meaningful for the broad class of networks, including those with time-delayed, noisy or mixed interactions.

In this paper several simple (random) graphs are studied illustrating the meaning of the concepts introduced in the paper.

Keywords: coupled oscillators, complex network, Möbius transformation, Kuramoto model, cross ratio

1. Introduction

Extensive research of large populations of coupled oscillators led to remarkable progress since 1975. and seminal paper of Kuramoto [1]. In some cases better understanding of dynamics and collective behavior of coupled oscillators unveiled unexpected relations to different mathematical theories. One idealistic model, for which the dynamics is particularly well understood, is a population of identical, globally (all-to-all) coupled phase oscillators with sinusoidal type of coupling function. In 1993. Watanabe and Strogatz [2, 3] reported transformation of variables that reduced dynamics of such population to low-dimensional submanifold. This was indication that such systems include some hidden symmetries and admit many constants of motion. Furthermore, in 2008, Ott and Antonsen [4, 5] reported new intriguing results for the case when initial phases of oscillators are uniformly distributed on $[0, 2\pi]$. For such initial data the evolution equation describing distribution of phases admits solutions belonging to particularly fine class (that are precisely Poisson kernels, see [6]) with reduction of dynamics to the submanifold of dimension two.[1] These reductions to low-dimensional dynamics are now well known as WS and OA ansatz respectively. In 2009. Marvel et al. [7] summarized these results and placed it in a broader mathematical context by explaining algebraic structure that stands behind them. They showed that such populations of oscillators dynamically generate automorphisms of unit disc as they are governed by the action of one-parametric family of Möbius transformations preserving the unit disc. There are many applications and consequences of this theoretical result (MMS principle[2]) that are still to be explored. In our point of view, the most important is in relating collective behavior of coupled oscillators to some fascinating mathematical objects that are extensively studied since XIX century up to today.

For this paper it is of key importance to clearly emphasize under what conditions a population of coupled oscillators obeys MMS principle. Firstly, oscillators are supposed to be identical, meaning that their intrinsic frequencies $\omega(t)$ are all equal. Second, each pair of oscillators is coupled with the same coupling strength $K(t)$. Third, the coupling function must be of sinusoidal type (sometimes also referred to as *Kuramoto-type coupling*), meaning that oscillators are coupled through the first harmonics only, and not through higher harmonics. Of course, these are very restrictive assumptions, especially the second one. One can not expect that such a fine algebraic structure (and low-dimensional behavior) will be found in any complex network of oscillators.[3] Nevertheless, in some papers OA ansatz is applied successfully to study complex networks of coupled oscillators [8, 9], heterogeneous [10], or hierarchically organized [11] populations of oscillators.

In this paper we characterize complex networks by detect-

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[1] OA ansatz applies also for nonidentical populations, for instance, if intrinsic frequencies of oscillators are chosen from Lorentzian distribution.

[2] We tend to refer to this result as WS ansatz (after Watanabe and Strogatz) or MMS principle (after Marvel, Mirollo and Strogatz).

[3] In fact, MMS principle can be slightly extended to some other idealistic models (say, network with two ideal clusters), as we will mention below.