Contents lists available at ScienceDirect

Physica D

journal homepage: www.elsevier.com/locate/physd

# Excitability, mixed-mode oscillations and transition to chaos in a stochastic ice ages model

D.V. Alexandrov *, I.A. Bashkirtseva, L.B. Ryashko

*Department of Theoretical and Mathematical Physics, Laboratory of Multi-Scale Mathematical Modeling, Ural Federal University, Lenin ave., 51, Ekaterinburg, 620000, Russian Federation*

## HIGHLIGHTS

- The article is devoted to excitability and chaotization in a 3D climate model.
- A limit cycle and stable equilibria represent the attractors of deterministic model.
- The formation of a mixed-mode regime of stochastic oscillations is revealed.
- A system transition from order to chaos with the increasing noise intensity is found.

## ARTICLE INFO

*Article history:*
Received 28 September 2015
Received in revised form
3 November 2016
Accepted 20 November 2016
Available online xxxx
Communicated by H.A. Dijkstra

*Keywords:*
Climatic model
Stochastic disturbances
Noise-induced transitions
Chaos

## ABSTRACT

Motivated by an important geophysical significance, we consider the influence of stochastic forcing on a simple three-dimensional climate model previously derived by Saltzman and Sutera. A nonlinear dynamical system governing three physical variables, the bulk ocean temperature, continental and marine ice masses, is analyzed in deterministic and stochastic cases. It is shown that the attractor of deterministic model is either a stable equilibrium or a limit cycle. We demonstrate that the process of continental ice melting occurs with a noise-dependent time delay as compared with marine ice melting. The paleoclimate cyclicity which is near 100 ky in a wide range of model parameters abruptly increases in the vicinity of a bifurcation point and depends on the noise intensity. In a zone of stable equilibria, the 3D climate model under consideration is extremely excitable. Even for a weak random noise, the stochastic trajectories demonstrate a transition from small- to large-amplitude stochastic oscillations (SLASO). In a zone of stable cycles, SLASO transitions are analyzed too. We show that such stochastic transitions play an important role in the formation of a mixed-mode paleoclimate scenario. This mixed-mode dynamics with the intermittency of large- and small-amplitude stochastic oscillations and coherence resonance are investigated via analysis of interspike intervals. A tendency of dynamic paleoclimate to abrupt and rapid glaciations and deglaciations as well as its transition from order to chaos with increasing noise are shown.

## 1. Introduction

The climate system comprises different physical and chemical processes freely interacting with the atmosphere and characterizing the natural rhythms of dynamic paleoclimatology. This includes interactions between the oceans, ice masses, anthropogenic and natural forcing (e.g., solar and Earth orbital radiation, changes in the atmospheric composition and temperature due to unpredictable volcanic eruptions). An important contribution comes from the astronomical forcing. It is well-known that changes in the Earth's orbit connected with astronomical variations have a strong impact on Earth's climate. So, for example, they are responsible for the glacial–interglacial cycles over the Quaternary (roughly the last 2.5 million years of Earth's history). An important point is that these glacial–interglacial interactions can be self-sustained [1], orbitally and stochastically forced [2,3] as well as possess their combination [4]. Note that Milankovitch cycles (variations in eccentricity, axial tilt, and precession of the Earth's orbit) are insufficient to explain the full range of Quaternary climate change, which also requires greenhouse gas and albedo variations, but they are a primary forcing that must be accounted for [5].

An important point of non-linear climate behavior is near 100 ky variations in the ice mass, atmospheric concentrations

---

* Corresponding author.
*E-mail address:* Dmitri.Alexandrov@urfu.ru (D.V. Alexandrov).

http://dx.doi.org/10.1016/j.physd.2016.11.007
0167-2789/© 2016 Elsevier B.V. All rights reserved.

Please cite this article in press as: D.V. Alexandrov, et al., Excitability, mixed-mode oscillations and transition to chaos in a stochastic ice ages model, Physica D (2016), http://dx.doi.org/10.1016/j.physd.2016.11.007

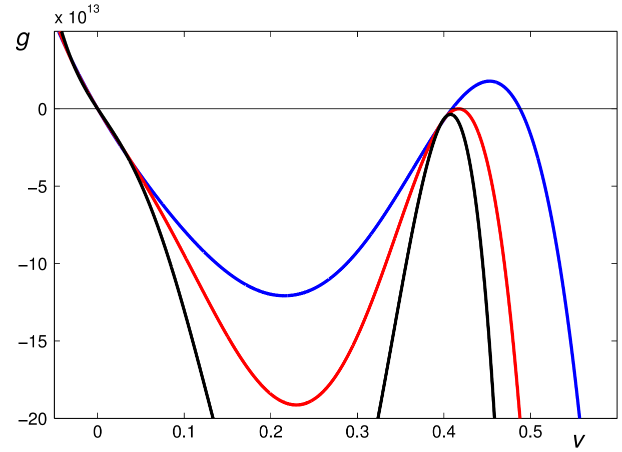

**Fig. 1.** Plots of $g(v)$ for various $\mu$ : $\mu = 0.1 \cdot 10^{-4}$ y$^{-1}$ (blue), $\mu = \mu_* = 0.1903 \cdot 10^{-4}$ y$^{-1}$ (red), and $\mu = 0.3 \cdot 10^{-4}$ y$^{-1}$ (black). The rest system parameters are [8,13]: $a_0 = 0$, $a_1 = 0.145$ y$^{-1}$, $a_2 = 1.86 \cdot 10^{-17}$ kg$^{-1}$, $a_3 = 1.265 \cdot 10^{14}$ y$^{-1}$, $b_0 = 0.276 \cdot 10^{-6}$ y$^{-1}$, $b_1 = 3.77 \cdot 10^{-4}$y$^{-1}$, $b_2 = 1.58 \cdot 10^{-18}$ kg$^{-1}$, $b_3 = 3.77 \cdot 10^{-34}$ kg$^{-2}$, $b_4 = 0.7152 \cdot 10^{-38}$ kg$^{-2}$, $b_5 = 0.697 \cdot 10^{13}$ y$^{-1}$, $c_0 = 0.792 \cdot 10^{-23}$ y$^{-1}$, $c_1 = 28.65 \cdot 10^{-23}$ y$^{-1}$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

of the greenhouse gases $CO_2$ and $CH_4$ and temperature [6–8] in the Quaternary period. These variations are responsible for the appearance of 100 ky ice age cycles [9], which, in particular, lead to water redistribution between oceans and continents, causing pressure changes in the upper mantle, with consequences for the melting of Earth's interior [10]. Note that this 100 ky period is close to the Earth's orbital eccentricity cycle [11]. Hand in hand with a dominant role of near 100 ky ice age cycles one can mention the ice mass fluctuations with smaller amplitudes and periods close to 20–40 ky [12]. These fluctuations can be connected with precessional orbital periods of the planet [13,14]. In addition, deglaciation periods are more rapid than the glacial buildups leading to a saw-tooth structures in the transient behavior of ice mass [8,13].

At present, there are a lot of mathematical models containing many free parameters that allow reasonable tuning of the model's outputs to observations. One of the main research tasks of these models within the conceptual framework consists in determination of possible variability mechanisms that govern the climate dynamics. Many authors bring them to the front using the dynamic systems theory [15–21]. In the present paper, we investigate such variational mechanisms induced by the nonlinearity and random fluctuations.

An analysis of stochastic and chaotic behavior of the climate feedback models represents a challenging problem of the modern nonlinear dynamics and mathematical modeling [22–26]. The first attempt to include different processes of stochastic forcing in the deterministic climate models was made by Saltzman with coauthors. Their pioneering works [27–29] on this subject are connected with simple computational modeling of stochastic (white noise) forcing in two-parametric (two-dimensional) climatic feedback systems. A more rigorous analysis of their two-dimensional stochastic model is recently carried out in [30,31], where the whole domain of system parameters is studied. However, up to now no detailed analysis was carried out on stochastic modeling in the three-dimensional climatic systems due to their complexity from the nonlinear dynamics point of view.

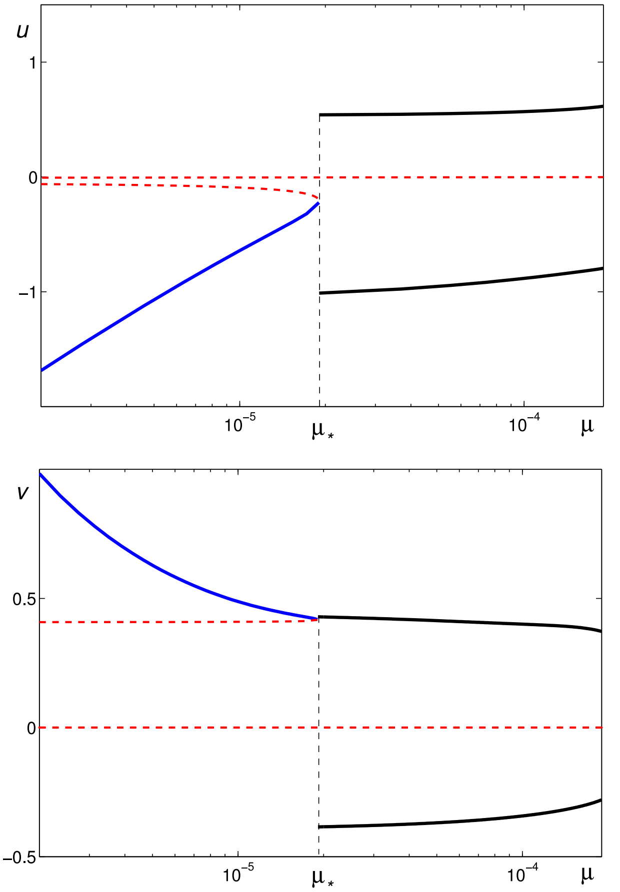

**Fig. 2.** Extrema of $u$- and $v$-coordinates of attractors of the deterministic system: stable equilibria (blue) and stable limit cycles (black). The unstable equilibria are shown by the red lines. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

A first three-dimensional deterministic model describing such glacial/deglacial variability has been derived and discussed by Saltzman and Sutera [13]. In this paper, we study their model to demonstrate a lot of new dynamic regimes connected with stochastic forcing as well as to argue in favor of new nonlinear phenomena in the dynamical paleoclimatology such as its chaotization and noise-induced excitability. This model represents a nonlinear dynamical system connecting three variables: continental ($\zeta$) and marine ($\chi$) ice masses, and bulk ocean temperature $\theta$. Let us briefly describe its main aspects that are of special physical significance. So, their model includes the following effects and processes [8,13]: (i) ice insulation, sea level – albedo, bedrock adjustment and meltwater – high latitude mixed layer temperature effects, (ii) ice – albedo, ice – baroclinicity and carbon dioxide feedbacks, (iii) marine ice buttressing of continental ice sheets and its inhibition of snowfall on interior of continental ice, (iv) ice stream forcing of shelf ice and its destruction by sea level rise and (v) gravitational collapse of ice sheets. Note that ignoring the astronomical forcing is a first step to describe the climate system dynamics. It is assumed that this climatic system has an equilibrium $(\hat{\zeta}, \hat{\chi}, \hat{\theta})$. Then, in the absence of deterministic and stochastic forcing, the deviations $(x, y, z) = (\zeta, \chi, \theta) - (\hat{\zeta}, \hat{\chi}, \hat{\theta})$ from this equilibrium can be described [8,13] by the following equations

$$\begin{aligned} \dot{x} &= a_0 x + a_1(1 - a_2 y)y - a_3 z \\ \dot{y} &= b_0 x + b_1(1 - b_2 y - b_3 y^2 - b_4 x^2)y - b_5 z \\ \dot{z} &= c_0 x + c_1 y - \mu z, \end{aligned} \tag{1}$$

where $\dot{x} = dx/dt$ and $t$ is the time. All coefficients $a_i$, $b_i$, $c_i$ and $\mu$ (which is the main system parameter determining the reverse relaxation time of the bulk ocean temperature to its mean value) are assumed to be positive constants.

In the present paper, we focus on the study of variability of dynamic regimes for this nonlinear model under the variations

Please cite this article in press as: D.V. Alexandrov, et al., Excitability, mixed-mode oscillations and transition to chaos in a stochastic ice ages model, Physica D (2016), http://dx.doi.org/10.1016/j.physd.2016.11.007