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Isolating blocks as computational tools in the circular restricted three-body problem

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HIGHLIGHTS

- An isolating block is defined around L_2 in the CRTBP using convexity conditions.
- A bisection method is used to compute invariant manifolds of planar Lyapunov orbits.
- A bisection method is used to compute invariant manifolds of the spatial invariant 3-sphere.
- Computed invariant manifolds are used to approximate the invariant 3-sphere.
- The low-energy transit portal into the Earth–Moon system is computed.

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ABSTRACT

Isolating blocks may be used as computational tools to search for the invariant manifolds of orbits and hyperbolic invariant sets associated with libration points while also giving additional insight into the dynamics of the flow in these regions. We use isolating blocks to investigate the dynamics of objects entering the Earth–Moon system in the circular restricted three-body problem with energies close to the energy of the L_2 libration point. Specifically, the stable and unstable manifolds of Lyapunov orbits and the hyperbolic invariant set around the libration points are obtained by numerically computing the way orbits exit from an isolating block in combination with a bisection method. Invariant spheres of solutions in the spatial problem may then be located using the resulting manifolds.

1. Introduction

It is well known from work by Conley, Easton, and McGehee that isolating blocks govern the behavior of transit orbits near the L_1 and L_2 libration points in the circular restricted three-body problem (CRTBP) [1–4]. This knowledge has been used in general to aid in visualizing the flow in these regions and specifically to assist in particular analyses such as the examination of heteroclinic connections between periodic orbits in the CRTBP found in Koon, Lo, Marsden, and Ross [5]. The L_2 gateway in particular is key for approach trajectories to a moon where multi-body effects are significant, a fact that was used in Anderson and Lo [6,7]. Ren and Shan [8,9] examined the boundary of orbits transiting through the gateway in a selected Poincaré section interior to the secondary using a bisection method and focused on using transit orbits as targets for optimization. They also noted that some known invariant manifolds fall on this boundary. Others have previously computed trajectories approximating libration orbit invariant manifolds by shooting trajectories toward either side of the libration point [10,11].

By making use of some of the fundamental characteristics of the isolating block, additional information and structures may be obtained. Specifically, the use of isolating blocks as a computational tool provides a means to both gain additional insight into the problem and compute the invariant manifolds of orbits and hyperbolic invariant sets associated with the libration points. The spatial CRTBP is used to model the dynamics of small masses in the Earth–Moon system, and this problem can be viewed as a three degree of freedom Hamiltonian system. The state space is six dimensional, and the manifolds of constant Jacobi energy have five dimensions. We investigate the dynamics of objects entering the Earth–Moon system with energies close to the energy of the L_2 Lagrange point (on the far side of the Moon). For the planar

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problem there are unstable periodic solutions near L_2, and the stable and unstable manifolds of these orbits guide the transition of orbits through the L_2 "bottleneck" formed when the Hill's region opens up. For the spatial problem with fixed Jacobi energy, this bottleneck contains a hyperbolic (unstable) invariant 3-sphere of solutions [12,13]. Stable and unstable manifolds of this invariant set guide the transit of orbits through the bottleneck just as in the planar problem.

Analytic methods for studying hyperbolic invariant sets in high dimensions are difficult to apply and are limited by issues of convergence in the normal form transformations [14,15] that are typically used. We introduce and use isolating blocks to locate invariant spheres of solutions and use a bisection method to numerically compute their stable and unstable manifolds. One virtue of the bisection method is that it requires relatively little analytic computation and can be implemented using numerical methods for solving ordinary differential equations. We choose arcs of initial conditions entering the block whose end points exit through disjoint exit sets. Every such arc must intersect the stable manifold of the trapped invariant set. Bisecting the arc and maintaining the end-point conditions leads to the accurate location of a point on the stable manifold. These points on the stable manifold may then be used to compute trajectories approaching the orbit or hyperbolic invariant set. Discarding an initial segment of an orbit on the stable manifold gives an approximation to its ω-limit set and therefore gives information about the dynamics restricted to the invariant set itself.

2. Circular restricted three-body problem equations of motion

A useful model for point mass or asteroid motion in the Earth–Moon system is the CRTBP in a rotating coordinate system with the Earth (or primary mass) located at position $E = (-\mu, 0, 0)$ and the Moon (or secondary mass) located at position $M = (\lambda, 0, 0)$ with $\lambda = 1 - \mu$. The mass parameter for the Earth–Moon system used here is $\mu = 1.2150584270571545 \times 10^{-2}$. The equations of motion for the point mass are

$$\begin{aligned} \ddot{x} &= \partial_x \Phi(x, y, z) - 2\dot{y} \\ \ddot{y} &= \partial_y \Phi(x, y, z) + 2\dot{x} \\ \ddot{z} &= \partial_z \Phi(x, y, z) \end{aligned} \tag{1}$$

where

$$\Phi(x, y, z) = \frac{1}{2}(x^2 + y^2) + U(x, y, z) \tag{2}$$

$$U(x, y, z) = \lambda/\rho_1(x, y, z) + \mu/\rho_2(x, y, z). \tag{3}$$

The functions $\rho_1(x, y, z)$ and $\rho_2(x, y, z)$ are the distances from the asteroid or spacecraft to the Earth and Moon, respectively.

It is sometimes convenient to use vector–matrix notation to represent these formulas. In that case, we use the notation $q = (x, y, z)'$ where $'$ indicates the transpose. Treating E, M, q, and $\nabla\Phi(q)$ as column vectors, we set $\rho_1 = |E - q|$ and $\rho_2 = |M - q|$. The equations of motion then take the form

$$\ddot{q} = \nabla\Phi(q) + 2A\dot{q} \tag{4}$$

$$\nabla\Phi(q) = Fq + \lambda(E - q)\rho_1{}^{-3} + \mu(M - q)\rho_2{}^{-3} \tag{5}$$

where

$$A = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad F = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{6}$$

The *Jacobi Integral* $J = \frac{1}{2}\langle \dot{q}, \dot{q}\rangle - \Phi(q)$ is a constant of motion for this model, and the *Jacobi constant*, C, is defined by the equation $J = -C/2$. This is a convenient choice since then $\langle \dot{q}, \dot{q}\rangle = 2\Phi(q) - C$. For positive values of C the feasible values of q form the *Hill's Region* defined by

$$m(C) = \{q : 2\Phi(q) - C \geq 0\}. \tag{7}$$

The proof that the Jacobi integral is constant on solutions is straightforward using this notation:

$$\dot{J} = \langle \ddot{q}, \dot{q}\rangle - \langle \nabla\Phi(q), \dot{q}\rangle = \langle 2A\dot{q}, \dot{q}\rangle = 0. \tag{8}$$

The Euler–Lagrange equilibrium points for the equations of motion are found by setting $\ddot{q} = \dot{q} = 0$.

As a reference, the Jacobi constants computed at each libration point in the Earth–Moon system are

$$\begin{aligned} C_{L1} &= 3.1883411054012485 \\ C_{L2} &= 3.1721604503998044 \\ C_{L3} &= 3.0121471493422489 \\ C_{L4} &= C_{L5} = 2.9879970524275450. \end{aligned} \tag{9}$$

The Hill's regions will be important in defining the isolating block, and the regions corresponding to several of these Jacobi constants are given in Fig. 1. It can be seen in these figures that as the Jacobi constant decreases, pathways open first at L_1 and then at L_2. These regions are important in defining the isolating blocks of interest for this study. See Pollard for a more detailed explanation of the CRTBP [16].

3. Computing invariant manifolds using isolating blocks

Isolating blocks have many theoretical uses in the study of dynamical systems [13,17,18]. However, their use as computational tools as discussed here may be new. The problem that suggests this use is the CRTBP in three space dimensions. Invariant three dimensional spheres of solutions are known to exist on five-dimensional energy surfaces with Jacobi constants close to those of the collinear Lagrange points. Later we will locate and investigate these spheres and their stable and unstable manifolds computationally.

For a smooth flow $\varphi(z, t)$ on a smooth manifold X contained in Euclidean space R^m, an *isolating block* is a compact subset B of X having continuous forward and backward exit time functions. The forward exit time function on B is defined by the formula

$$t^+(b) = \sup\{t : \varphi(b, s) \in B \text{ for } 0 \leq s \leq t\}. \tag{10}$$

An infinite forward exit time is allowed. The backward exit time function is similarly defined. Exit time functions are defined on compact sets, but they are discontinuous in general. Suppose for some time $0 < \sigma < t^+(b)$ it happens that $\varphi(b, \sigma) \in \partial B$. This is called an *internal tangency*, and initial points close to b may have exit times close to σ, and thus the exit time at b is discontinuous. If internal tangencies do not occur, then the exit time functions are continuous [2].

A useful condition that defines a block for a smooth flow on R^m is this: find a smooth real valued function V on R^m and a constant c such that $B = \{V \leq c\}$ is a compact manifold with boundary which is *convex to the flow*. This means that orbits that are tangent to a point z on the boundary of B "bounce off," or more precisely, there exists $\delta > 0$ such that $\varphi(z, t)$ does not belong to B provided $0 < |t| < \delta, \varphi(z, 0) = z$. The analytic condition that insures this convexity condition is that when

$$V = c \quad \text{and} \quad \dot{V} = 0, \quad \text{then } \ddot{V} > 0. \tag{11}$$

Numerical methods that approximate solutions of autonomous systems of ordinary differential equations use the discrete time dynamics of maps to approximate the "flows" of these systems. Time which flows (in theory) continuously in the original system is replaced by a discrete sequence of times. A parallel theory of isolating blocks for discrete time dynamics is available and is

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