ARTICLE IN PRESS

Physica D ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Contents lists available at ScienceDirect

Physica D

journal homepage: www.elsevier.com/locate/physd

Stability on time-dependent domains: convective and dilution effects

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HIGHLIGHTS

- Near-critical behavior of systems on time-dependent spatial domains is explored.
- Convective and dilution effects due to domain flow are studied.
- An amplitude equation governing pattern formation on time-dependent domains is derived.
- Phase slip phenomena are analyzed with both local and global stability analyses.
- A nonlinear phase equation describing the approach to a phase-slip event is derived.

ARTICLE INFO

Article history:
Received 5 February 2016
Accepted 20 October 2016
Available online xxxx
Communicated by Y. Nishiura

Keywords:
Stability theory
Eckhaus instability
Time-dependent domains
Amplitude equations
Pattern formation

ABSTRACT

We explore near-critical behavior of spatially extended systems on time-dependent spatial domains with convective and dilution effects due to domain *flow*. As a paradigm, we use the Swift–Hohenberg equation, which is the simplest nonlinear model with a non-zero critical wavenumber, to study dynamic pattern formation on time-dependent domains. A universal amplitude equation governing weakly nonlinear evolution of patterns on time-dependent domains is derived and proves to be a generalization of the standard Ginzburg–Landau equation. Its key solutions identified here demonstrate a substantial variety – spatially periodic states with a time-dependent wavenumber, steady spatially non-periodic states, and pulse-train solutions – in contrast to extended systems on time-fixed domains. The effects of domain flow, such as bifurcation delay due to domain growth and destabilization due to oscillatory domain flow, on the Eckhaus instability responsible for phase slips in spatially periodic states are analyzed with the help of both local and global stability analyses. A nonlinear phase equation describing the approach to a phase-slip event is derived. Detailed analysis of a phase slip using multiple time scale methods demonstrates different mechanisms governing the wavelength changing process at different stages.

1. Introduction

1.1. General setting

In the present work we consider pattern formation in a general evolution system on a time-dependent domain $\mathbf{x} \in \Omega_t$:

$$\frac{\partial c}{\partial t} + \nabla \cdot (\mathbf{u}\, c) = \mathcal{L}_{\mathbf{x}} c + N(c), \qquad N(0) = N'(0) = 0, \tag{1}$$

resulting from the application of a conservation law to a given quantity c such as a concentration (or concentrations if (1) is a vector equation). Here $\mathcal{L}_{\mathbf{x}}$ is a constant coefficient time-independent differential operator in the spatial variable $\mathbf{x}$ (in the case of a reaction–diffusion system $\mathcal{L}_{\mathbf{x}} = D\nabla^2$, for example); $N(c)$ is a general nonlinear differential operator, which may originate from the nonlinear part of the reaction law; $\mathbf{u}(t, \mathbf{x})$ is the velocity of a spatial domain point at location $\mathbf{x}$ at time t. The evolution of the quantity c is considered on a time-deforming domain Ω_t, which can be thought of as a 'substrate': examples include reaction–diffusion on growing skin, crown spike structure on a growing circular rim in the drop splash phenomenon, waves in a stretching rod, etc. [1]. The time-deformation of the domain Ω_t, which may be finite or infinite in spatial extent, introduces an advection term, $\mathbf{u} \cdot \nabla c$, corresponding to elementary volumes moving with the flow $\mathbf{u}(t, \mathbf{x})$ due to local domain deformation and a dilution term, $c\, \nabla \cdot \mathbf{u}$, corresponding to local volume change.

In addition to this Eulerian interpretation, it is instructive to look at the local flow $\mathbf{u}(t, \mathbf{x})$ from a Lagrangian description point of view. Suppose the point $\mathbf{x}$ in the domain Ω_t moves according

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http://dx.doi.org/10.1016/j.physd.2016.10.003
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to the trajectory $\mathbf{x} = \mathbf{X}(t, \mathbf{a}) \in \Omega_t$, where $\mathbf{a}$ is an initial position (point label or a Lagrangian coordinate), i.e. $\mathbf{X}(0, \mathbf{a}) = \mathbf{a}$. The local flow is then fully determined by $\mathbf{u}(t, \mathbf{x}) = \partial \mathbf{X}/\partial t$, where the partial derivative is evaluated at constant $\mathbf{a}$, i.e. a fixed marker for the given trajectory. We mention two key examples of domain deformation—for a more detailed discussion the reader is referred to [1]. First, *isotropic growth*, which in one spatial dimension corresponds to $X(t, a) = a\,r(t)$ with $r(0) = 1$ and $a \in [-L_0/2, L_0/2]$, implies

$$u(t, x) = a\,\dot{r} = x\frac{\dot{r}}{r} = x\frac{\dot{L}}{L} \equiv f(t)\,x, \tag{2}$$

where $L(t) = L_0 r(t)$ is the domain length at time t; the function $f(t) \equiv \dot{L}/L$ will be used throughout the rest of the paper. Thus the velocity u of stretching depends on the location away from the stationary center $x = 0$ with maximum speed at the domain boundaries $u|_{x=\pm L(t)/2} = \pm L_0\dot{r}/2 = \pm\dot{L}/2$. Note that the flow becomes a function of space only, $u = u(x)$, when the domain grows exponentially: $r \sim e^{\alpha t}$, $\alpha > 0$. Pattern formation on exponentially growing domains was studied numerically in a number of papers, e.g. [2–6]. If each point of the domain oscillates periodically with period T, then $X(t, a) = X(t + T, a)$ for all t and the flow $u(t, x)$ is called *oscillatory*—physically this may correspond to jittering endpoints of the domain (in opposite directions). When all points of the domain move with the same speed $u = u(t)$, the domain is simply *translated* and no stretching takes place—therefore, in what follows we only consider flows u depending on the spatial coordinate x (and perhaps time). An important special case is that of *uniform growth* of a spatially periodic domain studied in [7], corresponding to a one-dimensional domain that stretches at the same rate everywhere—because the domain does not have endpoints, each cross-section remains at rest and $u \equiv 0$ everywhere. This is the case, for example, during the process of crown formation in the drop splash problem [7], when the dilution effect is neglected.

1.2. Motivation and key questions

The present study is motivated by the need to understand domain flow (convection and dilution) effects on near-critical behavior and in particular on the Eckhaus instability in various physical systems. The Eckhaus instability is the key instability that permits a system exhibiting a periodic structure with a characteristic length scale to adjust to a growing domain by nucleating new wavelengths to maintain the characteristic scale of the pattern. To date, most knowledge in this area is based on experiments or numerical simulations, as illustrated by models of morphogenesis, e.g. [2–4,8,9]. Reaction–diffusion systems have been studied numerically on domains with both isotropic (including exponential) [2,4,6] and nonuniform [8] growth, revealing growth via wavelength-doubling. In [2] the flow term in Eq. (1) was neglected, thereby rendering the system analogous to translation-invariant systems with periodic boundary conditions [7]. Given the wealth of observational data, there is a need for a simple theory testing the basic mechanisms governing structure growth. One approach, based on a reduction of PDEs to ODEs, was proposed in [4] for domains with isotropic growth, but it does not seem to have the same clarity as the classical Eckhaus instability analysis [10]. Other known results about stability on time-dependent domains include the assertion that if the flow $\mathbf{u}$ is divergence-free the conditions for a diffusion-induced (Turing) instability remain unchanged from those for the time-independent case [9].

Given that fundamental understanding of stability properties on time-dependent domains is currently in a rudimentary state, in the present work we address basic questions centered around the effect of the flow $\mathbf{u}(t, \mathbf{x})$ on pattern formation in near-critical systems, in particular:

- Is there a universal near-critical amplitude equation similar to the Ginzburg–Landau equation (GLE) on time-independent spatial domains?
- How may the spatial structure of solutions be affected by time-evolution of the domain?
- What are the mechanisms by which the pattern wavelength (number of cells) changes and what is the nature of the boundary separating the basins of attraction of solutions with different number of cells as compared with the standard case of Eckhaus instability [11,12]?

The discussion below parallels the corresponding theory for patterns on time-independent domains [11,12] although the results demonstrate substantial differences arising from the presence of the domain flow $\mathbf{u}(t, \mathbf{x})$.

2. Amplitude equation

2.1. Problem statement and basic observations

We seek to understand the effects of domain growth on systems displaying spatially periodic structures. In reaction–diffusion systems of the form (1) this is only possible in coupled equations, i.e. when c is a vector of at least two concentrations [13]. We therefore turn to a generic scalar equation of fourth order exhibiting an intrinsic length scale, the Swift–Hohenberg equation (SHE)

$$c_t + (u\,c)_x = \mu\,c - (\partial_x^2 + k_0^2)^2\,c + N(c), \tag{3}$$

where $x \in [-L(t)/2,\ L(t)/2]$ and $L = L_0$ at $t = 0$. Fig. 1 shows the space–time evolution in the complex case with $N(c) = -|c|^2\,c$ resulting from an initial condition in the form of a stationary solution on a time-independent domain. The domain is assumed to be growing isotropically as described by Eq. (2). The figure reveals that new wavelengths are continuously injected in order to maintain a state with an intrinsic length scale of order $2\pi/k_0$. The advantage compared with a system of reaction–diffusion equations (1) is that both the bifurcation parameter μ and the intrinsic wavenumber k_0 appear explicitly, thereby making the derivation more compact without affecting the generality of the resulting amplitude equation—the same equation can be arrived at starting from a general near-critical system with $k_0 \neq 0$. In fact, the SHE (3) can be reduced to a vector reaction–diffusion form by introducing an auxiliary variable $c' = (\partial_x^2 + k_0^2)c$ as noted in [12], so that (1) becomes an algebraic–differential system in which the variable c' evolves on a much faster time scale compared to that of the master mode c.

Eq. (3) is considered on a finite domain whose length $L(t)$ is assumed to be large enough to contain many wavelengths $2\pi/k_0$ of the primary instability in the time-independent case. The wavenumber k_0 sets the intrinsic length scale of the problem, even though the primary instability leads to states with a time-dependent wavenumber $k(t)$, with $k(0) = k_0$ corresponding to $L(0) = L_0$ at $t = 0$—this will become clear once we set the details of the flow $u(t, x)$. As justified by the choice of the modulational scaling below, the departure of $k(t)$ from k_0 is assumed to be small and to evolve on slow time and spatial scales. Thus, without loss of generality, we 'delegate' this departure from k_0 to the modulational wavenumber $k(t) - k_0$, i.e. the wavenumber set by the solution of the amplitude equation.

Before deriving the amplitude equation, it is helpful to scale the spatial coordinate with respect to the domain size $L(t)$, $x \to L(t)\,x$. In the case of isotropic growth (2) we obtain

$$c_t = [\mu - f(t)]\,c - \left(\frac{1}{L^2(t)}\partial_x^2 + k_0^2\right)^2 c + N(c), \tag{4}$$

where $x \in [-1/2,\ 1/2]$. Thus isotropic domain growth has two main effects on the dynamics: it modifies the bifurcation