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Stability analysis of amplitude death in delay-coupled high-dimensional map networks and their design procedure

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HIGHLIGHTS

- Amplitude death in high-dimensional maps with delayed connections is analyzed.
- Several sufficient conditions for instability are obtained.
- Necessary conditions for stability are provided.
- These conditions and a concept of convex direction lead to a design procedure.
- These analytical results are confirmed numerically.

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ABSTRACT

The present paper studies amplitude death in high-dimensional maps coupled by time-delay connections. A linear stability analysis provides several sufficient conditions for an amplitude death state to be unstable, i.e., an odd number property and its extended properties. Furthermore, necessary conditions for stability are provided. These conditions, which reduce trial-and-error tasks for design, and the convex direction, which is a popular concept in the field of robust control, allow us to propose a design procedure for system parameters, such as coupling strength, connection delay, and input–output matrices, for a given network topology. These analytical results are confirmed numerically using delayed logistic maps, generalized Henon maps, and piecewise linear maps.

1. Introduction

A considerable number of studies have examined various phenomena in coupled continuous-time nonlinear oscillators [1–4] and coupled discrete-time nonlinear maps [5,6]. These phenomena are roughly classified into two types: weak- and strong-coupling induced phenomena. For weak coupling, the phases of coupled oscillators are governed by simple phase dynamics [7,8], and for strong coupling, their amplitudes are influenced by connections. Amplitude death, a phenomenon that occurs with strong coupling, has been widely investigated both analytically and experimentally [9,10]. This phenomenon is defined as a stabilization of unstable fixed points embedded within continuous-time nonlinear oscillators with diffusive connections. As this phenomenon can suppress oscillations, it has potential use in the avoidance of undesired oscillations for practical coupled systems [11–13]. However, diffusive connections, the most popular connections, never induce amplitude death in *identical* oscillators [14,15]. This fact is considered a drawback in terms of utilization of amplitude death.

It is well known that at least three types of connections can overcome this drawback: time-delay [16,17], dynamics [18–20], and conjugate connections [21]. Among these connections, there has been a gradual accumulation of analytical and experimental knowledge on *time-delay* induced amplitude death [16,17,22–25] because the transmission delays of information signals passing through connections [26,27] are ubiquitous in real situations. Many studies have examined time-delay induced death of coupled *continuous-time* oscillators [9,10]. Conversely, there have been several efforts to deal with amplitude death in coupled *discrete-time* maps.

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Time-delay-induced amplitude death of coupled *discrete-time* maps was reported in 2003 [28]. That study provided analytical results on death in *a pair of high-dimensional* maps with a delayed connection. The results are summarized as follows: (a) death cannot occur with no-delay connections; (b) the odd number property [15] exists even in a pair of high-dimensional maps; and (c) death cannot occur even with delay connections in a pair of *one*-dimensional maps. Result (b) was extended to *a simple ring lattice* [29]. Atay and Karabacak analytically investigated amplitude death in *one*-dimensional map *networks* with *uniform* delay time [30]. Masoller and Martí found amplitude death in one-dimensional map networks with *non-uniform* delay time [31], and the results were investigated analytically and numerically in detail [32–35]. However, few studies have attempted to deal with *high*-dimensional map *networks* [36] because it is not easy to analytically investigate their stability.

This study considers amplitude death in *high*-dimensional map *networks* with *uniform* delay time. We can deal with complex network topologies in the same manner as a simple topology. It is shown that the linear stability of amplitude death is governed by a characteristic equation with topology parameters. The characteristic equation reveals that results (a) and (b) in the previous study [28] for a pair of maps remain even for map networks. As the number of topology parameters is equivalent to that of maps in a network, one may think that a design of connection parameters in networks with a large number of maps is a complicated problem. However, we demonstrate that the convex direction [37], a strong mathematical concept for robust control theory, simplifies the design of connection parameters. We provide a systematic procedure that designs connection parameters (i.e., coupling strength and connection delay) and the input–output matrices of maps. Furthermore, result (b) is extended to reduce the number of trial-and-error tasks in the design procedure. The analytical results are confirmed numerically using three types of map networks, i.e., delayed logistic [38,39], generalized Henon [40], and piecewise linear [41,42] map networks. This paper is a substantially extended version of our previous conference paper [36].

2. Map networks

Consider the following high-dimensional maps,

$$\begin{cases} \boldsymbol{x}_i(n+1) = \boldsymbol{F}\left[\boldsymbol{x}_i(n)\right] + \boldsymbol{b}u_i(n), \\ \quad y_i(n) = \boldsymbol{c}\boldsymbol{x}_i(n), \end{cases} \qquad (i = 1, \ldots, N), \tag{1}$$

where $\boldsymbol{x}_i(n) \in \mathbb{R}^m$ is the system state of the ith m-dimensional map at time $n \in \mathbb{Z}$. The input and output signals are $u_i(n) \in \mathbb{R}$ and $y_i(n) \in \mathbb{R}$, respectively. $N \in \mathbb{Z}^+$ represents the number of maps. $\boldsymbol{F} : \mathbb{R}^m \to \mathbb{R}^m$ denotes the nonlinear map, which has at least one fixed point $\boldsymbol{x}^* : \boldsymbol{x}^* = \boldsymbol{F}\left[\boldsymbol{x}^*\right]$. The input and output matrices are $\boldsymbol{b} \in \mathbb{R}^m$ and $\boldsymbol{c} \in \mathbb{R}^{1\times m}$, respectively. Here, the input signal $u_i(n)$ with connection delay $\tau \in \mathbb{Z}^+$ and coupling strength $k \in \mathbb{R}$ is expressed as follows:

$$u_i(n) = k\left[\left\{\sum_{l=1}^{N}\frac{\varepsilon_{il}}{d_i}y_l(n-\tau)\right\} - y_i(n)\right], \tag{2}$$

$$d_i := \sum_{l=1}^{N}\varepsilon_{il},$$

where ε_{il} governs the network topology. Here, $\varepsilon_{il} = \varepsilon_{li} = 1(=0)$ indicates that maps i and l are (are not) connected. In addition, self-feedback is not allowed (i.e., $\varepsilon_{ii} = 0$). The number of maps connected to map i is expressed by d_i.

Here we focus on the following spatial uniform equilibrium state of map network (1) with connection delay (2):

$$\left[\boldsymbol{x}_1(n)^T \ \cdots \ \boldsymbol{x}_N(n)^T\right]^T = \left[\boldsymbol{x}^{*T} \ \cdots \ \boldsymbol{x}^{*T}\right]^T. \tag{3}$$

The linearized dynamics of a coupled map network (1) (2) at equilibrium state (3) is described as follows:

$$\boldsymbol{v}_i(n+1) = (\boldsymbol{A} - k\boldsymbol{b}\boldsymbol{c})\,\boldsymbol{v}_i(n) + k\boldsymbol{b}\boldsymbol{c}\sum_{l=1}^{N}\frac{\varepsilon_{il}}{d_i}\boldsymbol{v}_l(n-\tau), \tag{4}$$

$$\boldsymbol{A} := \left.\frac{\partial \boldsymbol{F}(\boldsymbol{x})}{\partial \boldsymbol{x}}\right|_{\boldsymbol{x}=\boldsymbol{x}^*}, \tag{5}$$

where $\boldsymbol{v}_i(n) := \boldsymbol{x}_i(n) - \boldsymbol{x}^*$. Here we employ the following assumption.

Assumption 1. The fixed point $\boldsymbol{x}^*$ of each isolated map is unstable, i.e., the Jacobi matrix $\boldsymbol{A}$ is unstable. Furthermore, $(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c})$ is assumed to be minimal.[1]

Linearized system (4) can be rewritten as follows:

$$\boldsymbol{V}(n+1) = \left[\boldsymbol{I}_N \otimes (\boldsymbol{A} - k\boldsymbol{b}\boldsymbol{c})\right]\boldsymbol{V}(n) + (\boldsymbol{E} \otimes k\boldsymbol{b}\boldsymbol{c})\boldsymbol{V}(n-\tau), \tag{6}$$

where $\boldsymbol{V}(n)$ and $\boldsymbol{E}$ are defined as

$$\boldsymbol{V}(n) := \begin{bmatrix} \boldsymbol{v}_1(n) \\ \vdots \\ \boldsymbol{v}_N(n) \end{bmatrix}, \qquad \boldsymbol{E} := \begin{bmatrix} \varepsilon_{11}/d_1 & \cdots & \varepsilon_{1N}/d_1 \\ \vdots & \ddots & \vdots \\ \varepsilon_{N1}/d_N & \cdots & \varepsilon_{NN}/d_N \end{bmatrix}.$$

Here the matrix $\boldsymbol{I}_N$ and the symbol $\otimes$ denote an N-dimensional identity matrix and the Kronecker product, respectively. Note that the stability of spatial uniform equilibrium state (3) is equal to that of the mN-dimensional linear system (6) with delay time τ. Substituting a solution $\boldsymbol{V}(n) = z^n\boldsymbol{a}$, with a nonzero vector $\boldsymbol{a} \in \mathbb{R}^{mN}$, into a linearized system (6) allows us to obtain its characteristic polynomial,

$$\bar{G}(z) := \det\left[z\boldsymbol{I}_{mN} - \boldsymbol{I}_N \otimes (\boldsymbol{A} - k\boldsymbol{b}\boldsymbol{c}) - (\boldsymbol{E} \otimes k\boldsymbol{b}\boldsymbol{c})z^{-\tau}\right], \tag{7}$$

which can be used to investigate the stability of equilibrium state (3). We can diagonalize the matrix $\boldsymbol{I}_N - \boldsymbol{E}$ using a matrix $\boldsymbol{T}$ [30,44] as follows:

$$\boldsymbol{T}^{-1}(\boldsymbol{I}_N - \boldsymbol{E})\boldsymbol{T} = \mathrm{diag}(\rho_1, \ldots, \rho_N). \tag{8}$$

Note that the eigenvalues of $\boldsymbol{I}_N - \boldsymbol{E}$, i.e., $\rho_i (i = 1, \ldots, N)$, satisfy

$$0 = \rho_1 \le \rho_2 \le \cdots \le \rho_N \le 2, \tag{9}$$

for any number of maps and topology [30]. This diagonalization simplifies characteristic polynomial (7), i.e.,

$$\bar{G}(z) := \prod_{q=1}^{N}\bar{g}(z, \rho_q), \tag{10}$$

$$\bar{g}(z, \rho) := d(z) + kn(z)\left\{1 - (1-\rho)z^{-\tau}\right\}. \tag{11}$$

Here $n(z)$ and $d(z)$ defined as

$$\frac{n(z)}{d(z)} := \boldsymbol{c}(z\boldsymbol{I}_m - \boldsymbol{A})^{-1}\boldsymbol{b} = \frac{\boldsymbol{c}\,\mathrm{adj}(z\boldsymbol{I}_m - \boldsymbol{A})\boldsymbol{b}}{\det\left[z\boldsymbol{I}_m - \boldsymbol{A}\right]}, \tag{12}$$

represent the transfer function of each map (1) from $u_i(n)$ to $y_i(n)$ around the fixed point $\boldsymbol{x}^*$. Note that $n(z)$ and $d(z)$ depend only on each isolated map (i.e., $\boldsymbol{A}$, $\boldsymbol{b}$, $\boldsymbol{c}$) but not on connection (2), i.e., k, ε_{il}, and τ. We summarize the above analytical argument as follows.

Lemma 1. *The local stability of spatial uniform equilibrium state* (3) *of map network* (1) (2) *is equivalent to the stability of the following*

[1] $(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c})$ is minimal if and only if they are controllable and observable [43]; they can be easily checked numerically.