Physica D 335 (2016) 45–53

Contents lists available at ScienceDirect

Physica D

journal homepage: www.elsevier.com/locate/physd

# Chaotic sub-dynamics in coupled logistic maps

Marek Lampart [a,b], Piotr Oprocha [c,b,*]

[a] Department of Applied Mathematics, VŠB - Technical University of Ostrava, 17. listopadu 15/2172, 708 33 Ostrava, Czech Republic

[b] IT4Innovations, VŠB - Technical University of Ostrava, 17. listopadu 15/2172, 708 33 Ostrava, Czech Republic

[c] AGH University of Science and Technology, Faculty of Applied Mathematics, al. A. Mickiewicza 30, 30-059 Kraków, Poland

ARTICLE INFO

*Article history:*
Accepted 17 June 2016
Available online 18 July 2016
Communicated by I. Melbourne

*Keywords:*
Coupled map lattices
Logistic map
Topological entropy
Attractor

ABSTRACT

We study the dynamics of Laplacian-type coupling induced by logistic family $f_\mu(x) = \mu x(1-x)$, where $\mu \in [0, 4]$, on a periodic lattice, that is the dynamics of maps of the form

$$F(x, y) = ((1-\varepsilon)f_\mu(x) + \varepsilon f_\mu(y), (1-\varepsilon)f_\mu(y) + \varepsilon f_\mu(x))$$

where $\varepsilon > 0$ determines strength of coupling. Our main objective is to analyze the structure of attractors in such systems and especially detect invariant regions with nontrivial dynamics outside the diagonal. In analytical way, we detect some regions of parameters for which a horseshoe is present; and using simulations global attractors and invariant sets are depicted.



## 1. Introduction

Many complex systems coming from applications are studied from the qualitative, long term dynamics point of view. In the last years, this approach was successful in considerations on states evolution in the coupled map lattices (CML), which are a class of models of extended media in which the relations between temporal evolution and spatial translation play a crucial role [1,2]. It has been observed that CML systems show variation of space–time patterns which are ordinary to other spatially enlarged systems (see,e.g., [3] or [4] and references therein).

If we consider one dimensional lattice, then there are various possibilities, and even for simply looking cases quite complex behavior can appear. If we browse the literature, then there are various concepts of communication between states. A simple, yet interesting, model, are one-way coupled logistic lattices which are modeled as a lattice $\{x^i\}_{i\in\mathbb{Z}} \subset [-1, 1]$ with evolution modeled by a discrete dynamical system ($n$ denotes values of lattice at $n$th iteration)

$$x^i_{n+1} = (1-\varepsilon)g(x^i_n) + \varepsilon g(x^{i-1}_n), \tag{1.1}$$

for the particular choice of map $g$, which is taken from logistic family $g_\mu(x) = 1 - \mu x^2$ for some $\mu$ (e.g. see [5,6]). Some other authors prefer to consider logistic family on $[0, 1]$, i.e. $f_\mu(x) = \mu x(1-x)$ where in most considered cases $\mu = 4$. Note that the dynamics of both interval maps $x \mapsto 1 - \mu x^2$ and $x \mapsto \mu x(1-x)$ is "similar", in the sense that they share most of interesting dynamical features [7]. In [8] the authors consider so-called two-ways communication (so-called Laplacian-type coupling)

$$x^i_{n+1} = (1-\varepsilon)g(x^i_n) + \frac{\varepsilon}{2}g(x^{i-1}_n) + \frac{\varepsilon}{2}g(x^{i+1}_n)$$

which is a restricted version of variable range coupling (e.g. see [9]):

$$x^i_{n+1} = (1-\varepsilon)g(x^i_n) + \frac{\varepsilon}{\eta(\alpha)}\sum_{j=1}^{N'}\frac{1}{j^\alpha}(g(x^{i-j}_n) + g(x^{i+j}_n))$$

where $x^i_n \in [0, 1]$ is the state variable for the site $i \in \{1, 2, \ldots, N\}$ at time $n$, $N > 0$ is odd, $\varepsilon \in [0, 1]$ is the coupling strength, $\alpha \in [0, \infty)$ is the effective range, and $\eta(\alpha) = 2\sum_{j=1}^{N'} j^{-\alpha}$ with $N' = (N-1)/2$ (see e.g. [10] and references therein). Simply, we assume that $\alpha$ is very large (say $\alpha \to \infty$) so coupling introduced by terms with $j \neq 1$ is not essential and can be ignored. Now, let us assume that initial values on the lattice are periodic, say $x^i_0 = x^j_0$ provided that $i = j \pmod 2$. Then we can view Laplacian-type

* Corresponding author at: AGH University of Science and Technology, Faculty of Applied Mathematics, al. A. Mickiewicza 30, 30-059 Kraków, Poland.

*E-mail addresses:* marek.lampart@vsb.cz (M. Lampart), oprocha@agh.edu.pl (P. Oprocha).

http://dx.doi.org/10.1016/j.physd.2016.06.010

coupling as a two dimensional map, since then

$$\begin{aligned} x_{n+1}^i &= (1-\varepsilon)g(x_n^i) + \frac{\varepsilon}{2}g(x_n^{i-1}) + \frac{\varepsilon}{2}g(x_n^{i+1}) \\ &= (1-\varepsilon)g(x_n^i) + \varepsilon g(x_n^{i+1}) \\ &= (1-\varepsilon)g(x_n^i) + \varepsilon g(x_n^{i-1}). \end{aligned} \tag{1.2}$$

In this paper we will consider the above mentioned situation. It is also clear that instead of 2 variables we can consider $n$ variables, simply assuming $n$-periodicity instead of 2-periodicity on the lattice (also other type of coupling can be considered in that case).

The model (1.2) was considered by many authors from different points of view in the last 30 years. Many interesting results were revealed, however most of them were obtained as a result of numerical simulations and much smaller insight was done using analytic methods. Spectral properties of the model (1.2) generalizations were discussed in [11] and conditions for the stability of spatially homogeneous chaotic solutions were presented. Some studies whether there is a synchronization were undertaken in [12] and later, e.g., in [13,14] or [15] (for more comments, see references therein). The existence of wavelike solution in the model (1.2) for which the spatiotemporal periodic pattern can be predicted was investigated in [16] (see also references therein). The bifurcation phenomena and loss of synchronization were examined in [17,5]. It was also observed that the model exhibits periodic structure, multiple attractors, entrained and phase reversed patterns as well as chaos, e.g., see [18,19]. The model (1.2) was also analyzed for negative coupling constant revealing existence of a forward invariant curve [20].

In [21] the authors proved a few basic facts on the model (1.2) including the existence of chaos in the sense of Li and Yorke for some range of parameters. It is very easy to see that the model exhibits chaotic behavior for zero coupling constant under the condition that $f_\mu$ is chaotic, since in this case the model is acting just as a Cartesian product of chaotic $f_\mu$. There is also always an invariant diagonal, where dynamics is the same as $f_\mu$. In this direction a natural question arise, whether large off-diagonal regions of chaotic dynamics could be observed for non-zero coupling constant. One of our aims is to detect regions outside the diagonal, where chaotic dynamics can be supported.

A natural approach is to split the region of parameters into several areas. For the first area all points are attracted to the main invariant subsystem, which is always embedded in the diagonal (see Theorem 3), and for the second case the model is showing chaotic motion outside the diagonal, i.e., there is a horseshoe located outside the diagonal (see Theorem 8). A sharp edge between these two opposite situations is yet to be determined. As we announced earlier, the study is focused on the two dimensional case (i.e. lattice with 2-periodic entries), since in higher dimension (i.e., larger period on the lattice) the problem could be dealt with similarly by analogous calculations and arguments.

The paper is organized as follows. In Section 2 the model is presented, in Section 3 parameters which lead to the diagonal as an attractor are characterized, and finally in Section 4 nontrivial horseshoes (and invariant subsets) are detected. Appendix at the end of the paper contains description of all fixed points in the model.

## 2. The model

Let us define the family of *logistic maps* $f_\mu : [0, 1] \to [0, 1]$, where $0 \le \mu \le 4$, by

$$f_\mu(x) = \mu x(1-x).$$

The interval $I_\mu = [f_\mu^2(1/2), f_\mu(1/2)]$ is called *the core of* $f_\mu$, when $\mu \in (2, 4]$, see Fig. 1. For the choice of parameter $\mu \in [0, 2]$ the interval $I_\mu$ is still well defined but does not have such nice properties. Namely, $I_2 = \{1/2\}, I_0 = \{0\}$ and for $\mu \in (0, 2)$ it is not invariant under $f_\mu$, hence not remarkable in these cases. The core $I_\mu$ is strongly invariant, that is $f_\mu(I_\mu) = I_\mu$, and every point from $(0, 1)$ is attracted to $I_\mu$. The dynamics on the core can be very rich. For example, in [22] the authors show that for the family of tent maps the dynamics on the core is topologically exact for some range of parameters, which, generally speaking, means that most rich dynamical behavior is present in the core. In the case of logistic maps, the calculations are much harder and spectrum of possible dynamical behaviors is richer. However, it is known that for some parameters the dynamics on the core of logistic map is the same (in the sense of topological conjugacy) as on the core of tent map with slope corresponding to $\mu$ (e.g. see [7]).

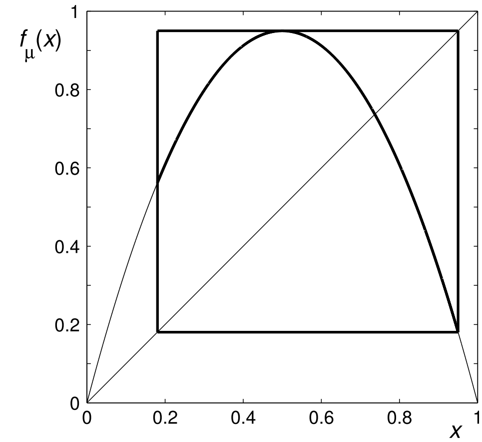


**Fig. 1.** Graph of $f_\mu$ for $\mu = 3.8$ and the graph restricted to the core $I_\mu$ (bounded by box).

By *coupled logistic map (with coupling constant* $\varepsilon$*)* the following map of the square $F: [0, 1]^2 \to [0, 1]^2$ is meant,

$$F(x, y) = (F_x(x, y), F_y(x, y)) \tag{2.1}$$

where

$$F_x(x, y) = (1-\varepsilon)f_\mu(x) + \varepsilon f_\mu(y), \tag{2.2}$$

$$F_y(x, y) = (1-\varepsilon)f_\mu(y) + \varepsilon f_\mu(x). \tag{2.3}$$

Simple calculations yield that $\mathrm{Fix}(F) \supseteq \{(0, 0), (p_\mu, p_\mu)\}$ for $\mu > 1$, where $p_\mu = (\mu - 1)/\mu$ and they are the only fixed points of $F$ on the diagonal. Note that the point $p_\mu$ is fixed for $f_\mu$ and is attracting or repulsive for $\mu \in (1, 3)$ or $\mu \in (3, 4]$, respectively. It may happen that $\mathrm{Fix}(F)$ contains also points outside diagonal, however always $\#\mathrm{Fix}(F) \le 4$. For the reader convenience we present formulas for all fixed points of $F$ and their dependence on parameters $\mu, \varepsilon$ in the Appendix. The periodic structure of $F$ was also studied by [23] (for more see references therein), where fixed points as well as two-cycles are listed with respect to the diagonal.

Here, as usual, $\mathrm{Fix}(F)$ and $\mathrm{Per}(F)$ stand for the set of all fixed and periodic points of $F$, respectively; and the space $\mathbb{R}^2$ is endowed with the Euclidean norm $\|\cdot\|$.

**Remark 1.** It is clear that $\Delta = \{(x, x) : x \in [0, 1]\}$ is an invariant subset for $F$ and that $F|_\Delta$ can be identified with $f_\mu$ by natural homeomorphism $\pi: [0, 1] \ni x \mapsto (x, x) \in \Delta$, that is $F|_\Delta \circ \pi = \pi \circ f_\mu$ or equivalently $F|_\Delta = \pi \circ f_\mu \circ \pi^{-1}$.

By Remark 1 it is clear that we have a lower bound for topological entropy of $F$ given by $h_{\mathrm{top}}(f_\mu) \le h_{\mathrm{top}}(F)$, $F$ is chaotic in the sense of Li and Yorke, provided that $f_\mu$ is chaotic in the sense of Li and Yorke, etc. The reader not familiar with theory of entropy is referred to books [24] or [25]. Many recent advances on chaos in the sense of Li and Yorke can be found in [26].