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# Timing Variation in an Analytically Solvable Chaotic System

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**Abstract**

We present analytic solutions for a chaotic dynamical system that do not have the regular timing characteristic of recently reported solvable chaotic systems. The dynamical system can be viewed as a first order filter with binary feedback. The feedback state may be switched only at instants defined by an external clock signal. Generalizing from a period one clock, we show analytic solutions for period two and higher period clocks. We show that even when the clock 'ticks' randomly the chaotic system has an analytic solution. These solutions can be visualized in a stroboscopic map whose complexity increases with the complexity of the clock. We provide both analytic results as well as experimental data from an electronic circuit implementation of the system. Our findings bridge the gap between the irregular timing of well known chaotic systems such as Lorenz and Rossler and the well regulated oscillations of recently reported solvable chaotic systems.

*Keywords:* chaos, analytic solution, electronic circuit, filter, return map

## 1. Introduction

Analytic solutions to chaotic dynamical systems are few and far between [1, 2]. Without question, the most common approach to solving chaotic equations is numerical approximation [3, 4]. However, some hybrid chaotic systems have been introduced that have analytic solutions [5, 6, 7, 8]. These systems violate the conventional wisdom that chaos must be studied using computational methods. Important theoretical quantities, such as Lyapunov exponents and metric entropy, that can usually only be estimated numerically can be determined analytically. Some examples are conjugate to symbolic dynamical systems for which the existence of chaos is known rigorously. Several different attractor topologies have been found.

A striking feature of all of these solvable systems is the regular timing [9]. For example, the second order system solution has regular zero crossings. This regular timing is particularly attractive from the point of view of applications such as radar, sonar, and communications [10, 11, 12, 13, 14, 15, 16]. In these technologies it is usually desirable to regulate the timing of transmitted and received signals according to a system clock. Thus, chaotic solutions with regular zero crossings, for example, can be easily integrated into conventional technology. However, regular timing may also be viewed as somewhat artificial. Familiar examples of chaos do not have regular timing [17, 18, 19]. Thus, it is desirable to know whether there are dynamical systems that bridge the gap between solvable hybrid systems and familiar chaotic oscillators. An effort to do just that is the subject of this paper.

Recently, solvable chaos was demonstrated in a first order filter whose set point is updated at regular intervals [8]. Unlike previously studied second order hybrid systems whose update times were based on their own natural frequency, the regular timing had to be imposed externally in this case. In this paper, we take advantage of the separation of the timing from the oscillation in the first order system to explore more complex periodic timing signals, and even irregular timing signals. We show that the system remains solvable despite this timing variation. Furthermore, we observe how the structure of the solutions changes as the timing becomes more irregular. Our theoretical and numerical results are confirmed through observations of an electronic implementation of the first order filter with a switched set point.

## 2. Analytically Solvable Chaotic Oscillator Based on a First-Order Filter

Previously, it was shown that an unstable first order filter whose set point is regularly updated according to the timing of an external clock is chaotic and has an analytic solution [8]. The filter is described by the linear differential equation

$$\frac{du}{dt} = u - s \tag{1}$$

where $u(t) \in \mathbb{R}$ is a continuous state and the set point $s(t) \in \{-1, 1\}$ is a discrete state. The discrete state is updated at instants defined by an external clock. Regular

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