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Arnold's mechanism of diffusion in the spatial circular restricted three-body problem: A semi-analytical argument

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HIGHLIGHTS

- We study the dynamics of the spatial circular restricted three-body problem near L_1.
- We describe a mechanism for Hamiltonian instability (Arnold diffusion).
- We construct orbits that 'diffuse' across a 3-sphere near L_1.
- These orbits change significantly the amplitude of motion relative to the ecliptic.
- We provide analytical results and numerical evidence for diffusing orbits.

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ABSTRACT

We consider the spatial circular restricted three-body problem, on the motion of an infinitesimal body under the gravity of Sun and Earth. This can be described by a 3-degree of freedom Hamiltonian system. We fix an energy level close to that of the collinear libration point L_1, located between Sun and Earth. Near L_1 there exists a normally hyperbolic invariant manifold, diffeomorphic to a 3-sphere. For an orbit confined to this 3-sphere, the amplitude of the motion relative to the ecliptic (the plane of the orbits of Sun and Earth) can vary only slightly.

We show that we can obtain new orbits whose amplitude of motion relative to the ecliptic changes significantly, by following orbits of the flow restricted to the 3-sphere alternatively with homoclinic orbits that turn around the Earth. We provide an abstract theorem for the existence of such 'diffusing' orbits, and numerical evidence that the premises of the theorem are satisfied in the three-body problem considered here. We provide an explicit construction of diffusing orbits.

The geometric mechanism underlying this construction is reminiscent of the Arnold diffusion problem for Hamiltonian systems. Our argument, however, does not involve transition chains of tori as in the classical example of Arnold. We exploit mostly the 'outer dynamics' along homoclinic orbits, and use very little information on the 'inner dynamics' restricted to the 3-sphere.

As a possible application to astrodynamics, diffusing orbits as above can be used to design low cost maneuvers to change the inclination of an orbit of a satellite near L_1 from a nearly-planar orbit to a tilted orbit with respect to the ecliptic. We explore different energy levels, and estimate the largest orbital inclination that can be achieved through our construction.

1. Introduction

1. *The instability problem in Hamiltonian dynamics*. Many physical systems that conserve mechanical energy can be modeled as (autonomous) Hamiltonian systems. Typical Hamiltonian systems exhibit chaotic dynamics. Integrable Hamiltonian systems – those that can be solved by quadratures – are rare. In applications, one

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frequently encounters *nearly integrable Hamiltonian systems*, which are small perturbations of integrable ones. Studying the long-term behavior of nearly integrable Hamiltonian systems was considered by Poincaré to be *the fundamental problem of dynamics*.

An underlying question is whether the effect of small perturbations averages out in the long run, or it can accumulate to large effects. Arnold [1] conjectured that, for 'typical' nearly-integrable systems, there always exist orbits that '*diffuse*', i.e., they travel a large distance in the phase space, for all sufficiently small perturbations. A brief overview on Arnold's diffusion problem and on recent progress is given in Section 6 below. Many results on the Arnold diffusion problem concern 'generic' classes of Hamiltonian systems. There are only few concrete examples in the literature where diffusing orbits can be found explicitly.

2. *Model and main results*. In this paper we describe a simple model from celestial mechanics which exhibits diffusing orbits.

We also describe a general method, relying on geometric and topological tools, that can be used to find explicitly such orbits. This method can be applied to other models not considered in this paper (see. e.g. [2]).

We consider the spatial circular restricted three-body problem (SCRTBP), in the case of the Sun–Earth system. This models the motion of an infinitesimal particle (satellite) relative to two massive bodies (Sun and Earth), which are assumed to move on circular orbits about their center of mass. In the spatial problem the infinitesimal particle is not constrained to move in the same plane as the massive bodies (the ecliptic plane).

This system can be described by a 3-degree of freedom Hamiltonian system. There are five equilibrium points for the Hamiltonian flow: three of them are of center–center–saddle type, and the other two are of center–center–center type. In this paper we focus on one of the center–center–saddle equilibria, referred to as L_1, which is located between Sun and Earth. (We point out that we can carry a similar analysis on the other two equilibria of center–center–saddle type.)

The dynamics near L_1 is organized by some remarkable geometric objects. For an energy level close to that of L_1, the flow restricted to the energy manifold has a 3-dimensional normally hyperbolic invariant manifold (NHIM) near L_1.

The NHIM is diffeomorphic to a 3-sphere, and contains many 2-dimensional invariant tori that are closely spaced. (The existence of families of 2-dimensional invariant tori as above can be established, under certain restrictions on the parameters of the model, by using the KAM theorem.) Thus, any orbit of the '*inner dynamics*' – the restriction of the flow to the 3-sphere – is either confined to an invariant torus, or to one of the narrow '*gaps*' between two invariant 2-tori that separate the 3-sphere. For each orbit lying on such a torus the out-of plane amplitude relative to the ecliptic is fixed, and for each orbit lying inside a gap the out-of plane amplitude varies only very little. Thus, there are no orbits for the inner dynamics that can 'diffuse' across the sphere; in other words, it is impossible to achieve a large change of the out-of-plane amplitude of an orbit by using only the inner dynamics. In this sense, the inner dynamics is similar to that of an integrable system.

In order to obtain orbits that diffuse across the sphere one has to use also the '*outer dynamics*'—along homoclinic orbits bi-asymptotic to the 3-sphere. As it turns out, the stable and unstable manifolds of the 3-sphere intersect transversally, yielding multiple homoclinic manifolds, which consist of smooth families of homoclinic orbits. By following carefully selected homoclinic orbits one can increase/decrease the out-of-plane amplitude by an amount that is larger than the size of the gaps between the invariant tori. We construct pseudo-orbits – obtained by repeatedly intertwining the outer dynamics with the inner dynamics – that *change the out-of-plane amplitude by a 'large amount'*. We then show, via a *topological version of the shadowing lemma*, that there exist true orbits that follow closely those pseudo-orbits.

Our methodology combines analytical results with numerical methods. The main analytical result is an abstract theorem that provides the existence of diffusing orbits under verifiable conditions. The numerical part consists of verifying the conditions of the theorem, and of explicitly detecting diffusing orbits through careful computations. More details are given in Section 4 below.

3. *Applications*. A possible application of this mechanism is to design low cost procedures that change the out-of-plane amplitude, relative to the ecliptic, of the motion of a satellite near L_1, from nearly zero amplitude to some 'large' amplitude. We also provide practical information on how to choose the energy level of the system in order to obtain such motions that end up near the largest possible out-of-plane amplitude for that energy level.

Our focus on L_1 can be viewed as a 'proof of concept'. Similar type of orbits can be designed near the other equilibrium points of center–center–saddle type.

We also point out that in this paper we consider a relatively narrow range of energies near that of L_1. At higher energies, new types of orbits appear – the so called halo orbits –, which are rather useful for space mission design. In practice, it is not too difficult to jump to the halo orbits (see [3]). At higher energies, the dynamics restricted to the 3-sphere does no longer resemble that of an nearly-integrable system—the Poincaré section reveals large 'elliptic islands' and a large 'stochastic sea'. Hence, the underlying dynamics is not nearly integrable, as in the case of the Arnold diffusion problem, which is one of the motivations of our work. In fact, the closer the energy level is to that of L_1, the closer the dynamics on the 3-sphere is to that of an integrable system, hence the more difficult is to achieve diffusion. So in this paper we deliberately choose to consider a more difficult problem.

4. *Methodology*. In his original paper [1], Arnold described a mechanism of diffusion based on transition chains consisting of KAM tori and transverse heteroclinic connections among them. Such transition chains were also used in other papers, e.g., [4–10]. Other mechanisms of diffusion make use of transition chains of secondary tori [8], or of Aubry-Mather sets [11–13].

In comparison to the works mentioned above, in this paper we do not use transition chains formed by invariant tori or by Aubry Mather sets. Instead, we parameterize the 3-sphere near L_1 by a system of coordinates consisting of one action and two angle variables, – with the action variable corresponding to the out-of-plane amplitude of the orbit about the ecliptic –, and form transition chains of level sets of the action. These level sets are not necessarily invariant sets, but only 'almost invariant'. Once we obtain a parametrization of the 3-sphere by action–angle variables, computing level sets of the action is trivial. Consequently, our method is computationally very cheap. In contrast, the precise computation of KAM tori, secondary tori, or Aubry-Mather sets is more laborious.

We mention that transition chains of level sets of the action also appear in [14], but they have not been numerically implemented before.

Now we briefly explain the construction of transition chains of action level sets. The main tool is a geometrically defined mapping on the NHIM, referred to as the scattering map (see [15]). For a fixed choice of a homoclinic manifold, follow an unstable fiber whose foot point lies on the NHIM up to the homoclinic manifold; then identify a stable fiber that passes through the same homoclinic point, and follow the stable fiber up to its foot point on the NHIM; the mapping that assigns to the foot point of the unstable fiber the foot point of the stable fiber is the scattering map. In order for this map to be well defined, one has to fix a suitable restriction of the homoclinic manifold. The resulting scattering map is in general defined only on some open subset of the NHIM.

The scattering map provides a convenient way to track the effect on the action variable of following homoclinic orbits.