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Chaos near a resonant inclination-flip

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H I G H L I G H T S

- A new model of chaotic dynamics is described.
- The model is realized in an attractor of a flow near a resonant inclination-flip orbit.
- Tools in computational topology and dynamics are used to characterize the dynamics.

A R T I C L E   I N F O

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A B S T R A C T

Horseshoes play a central role in dynamical systems and are observed in many chaotic systems. However most points in a neighborhood of the horseshoe escape after finitely many iterations. In this work we construct a new model by re-injecting the points that escape the horseshoe. We show that this model can be realized within an attractor of a flow arising from a three-dimensional vector field, after perturbation of an inclination-flip homoclinic orbit with a resonance. The dynamics of this model, without considering the re-injection, often contains a cuspidal horseshoe with positive entropy, and we show that for a computational example the dynamics with re-injection can have more complexity than the cuspidal horseshoe alone.

1. Introduction

Smale's horseshoe map was introduced as an example of a chaotic and hyperbolic dynamical system which is topologically transitive and contains a countable set of periodic orbits [1]. The maximal invariant set of this map, the horseshoe, is the topological product of two Cantor sets, and the dynamics on this set is chaotic. The complement of the horseshoe is dense in a neighborhood so that nearby points escape after finitely many iterates. This model is observed when the stable manifold and the unstable manifold of a hyperbolic fixed point of a two-dimensional diffeomorphism intersect transversally [2]. This type of dynamics is also observed in the Poincaré return map of some three-dimensional vector fields, see the discussion below.

In this article we propose a model that contains similar features, but also dynamics with more complexity. This model is essentially obtained by *re-injecting* the points that escape a neighborhood of the horseshoe. We show that this model can be realized by considering the unfolding of a three-dimensional vector field that possesses two homoclinic orbits to the same resonant, hyperbolic equilibrium point, one of which is degenerate and the other is nondegenerate.

Homoclinic orbits can play an important role in the dynamics of flows; the corresponding dynamics is structurally unstable and therefore may lead to dramatic changes in the dynamics. The complexity of the dynamics obtained after perturbation is often related to the degeneracy of the unperturbed system, i.e. the number or parameters that are necessary for a typical unfolding.

Degenerate dynamics involving homoclinic orbits have received a lot of interest over the last few decades. Shil'nikov shows the presence of chaos near a homoclinic orbit when the linearization matrix at the saddle point has complex eigenvalues under certain conditions, see [3]. When the linearization has three real eigenvalues, Deng [4] shows that the unfolding of a degenerate, *critically twisted*, homoclinic orbit can lead to a suspended horseshoe. Deng describes a bifurcation scenario where the horseshoe is destroyed (or created) after infinitely many homoclinic bifurcations. Following this scenario, chaotic dynamics are observed, and the corresponding invariant set is called a *cuspidal horseshoe*, described in Fig. 1. It is shown that a suspended cuspidal horseshoe can be realized in the unfolding of a degenerate homoclinic orbit in $\mathbb{R}^3$. In [4], critically twisted can mean two possible configurations—

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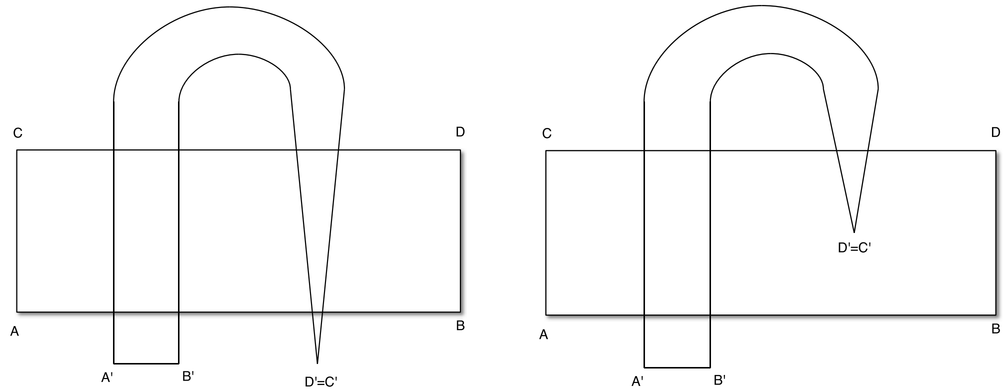

Fig. 1. Two possible configurations of a cuspidal horseshoe. On the left, the dynamics is a full shift on two symbols. On the right, there is only a partial shift, which may or may not be chaotic.

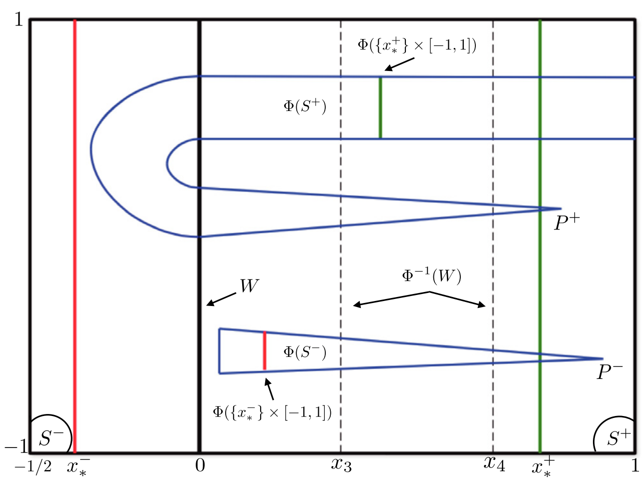

Fig. 2. Description of the map Φ.

the *orbit-flip* and the *inclination-flip*, see below for more details and definitions.

In [5], the authors show that the scenario presented by Deng is possible in the case of an inclination-flip homoclinic orbit, as long as the unperturbed system satisfies some open condition. The authors study the Poincaré return map Ψ on a cross section. For typical values of the parameters, the corresponding dynamics generalize that of Smale's horseshoe. Restricted to the maximal invariant set Λ, the Poincaré return map is conjugate to a partial shift on two symbols i.e., there exists a set $\mathcal{B} \subset \{0,1\}^{\mathbb{Z}}$ that is invariant under the shift $\sigma : \{0,1\}^{\mathbb{Z}} \to \{0,1\}^{\mathbb{Z}}$ and a homeomorphism $\xi : \mathcal{B} \to \Lambda$ such that

$$\Psi \circ \xi = \xi \circ \sigma.$$

Similar results are obtained in [6] in the case of the orbit-flip. Note that in general, the corresponding dynamics is not a subshift of finite type [5,6].

The unfolding of degenerate homoclinic orbits can lead to even more complicated dynamics. A Hénon like attractor can be observed in the unfolding of a degenerate homoclinic orbit [7,8], and Lorenz attractors can be observed after perturbation of a vector-field having a pair of homoclinic orbits [9,10]. When studying a homoclinic orbit to a hyperbolic saddle, one often encounters the difficulty in estimating the *Dulac map*, that is the transition map between a section transverse to the local stable manifold to a section transverse to the unstable one. The presence of resonance has often been related with the lack of smoothness of the linearization near the singularity, and therefore making the estimation of the Dulac map more complicated. However, it can lead to increasing the complexity of the dynamics, see for instance [9,11] for more details.

The paper is organized as follows. In the remainder of the introduction, we present a model of a two-dimensional map on the plane which characterizes a *re-injected horseshoe map*. An example of an explicit map that satisfies the model properties can be found in [12]. We also state Theorem 1, the main result of the article, which realizes a re-injected horseshoe in Poincaré maps of a family of three-dimensional vector fields, and we give an outline of the steps in the proof. The proof of Theorem 1 is then given in Sections 2 and 3. In the final section we study an explicit map by choosing parameters for an approximate Poincaré map. We give a rigorous, computer-assisted characterization of the dynamics on the maximal invariant set of this map, which acts like a re-injected horseshoe. This characterization is through a semiconjugacy to a symbolic system. We then compare this to the dynamics of the corresponding cuspidal horseshoe alone, and the results strongly suggest that the dynamics of the full system has more complexity than that of the cuspidal horseshoe without re-injection.

1.1. The model of the re-injected horseshoe map

In the (x, y)-plane consider the domain

$$S = S^+ \cup S^- \quad \text{where } S^+ = (0,1] \times [-1,1], \quad \text{and} \quad S^- = [-1/2, 0) \times [-1,1].$$

The image of S under Φ is described in Fig. 2. The restriction of Φ to S^+ is like a horseshoe except that one boundary is collapsed onto a cusp. The points that escape S^+ are re-injected into S^+ via the action of Φ on S^-. More precisely the model satisfies the following properties:

(i) Φ maps S^+ and S^- diffeomorphically onto their respective images and $\Phi(S^-) \subset S^+$,

(ii) for each $y \in [-1,1]$, $\Phi((0,1] \times \{y\})$ is a C^1 curve that intersects $W = \{0\} \times [-1,1]$ exactly twice,

(iii) there exists $0 < x_3 < x_4 < 1$ such that

$$\Phi(\{x_3\} \times [-1,1]) \subset W, \qquad \Phi(\{x_4\} \times [-1,1]) \subset W,$$
$$\text{and} \quad \Phi(\{x\} \times [-1,1]) \cap W = \emptyset \text{ for all } x \neq x_3, x_4,$$

(iv) for each $x \in [-1/2, 0) \cup (0, x_3) \cup (x_4, 1]$, there exists $-1/2 < L_x < 1$ such that $\Phi(\{x\} \times [-1,1]) \subset \{L_x\} \times [-1,1]$ so that vertical line segments form a Φ-invariant foliation of $S^+ \cup S^-$. Moreover, the map is expanding on this foliation, i.e. for any pair x, x' with $x \neq x'$ which are both contained in $[-1/2, 0)$, both contained in $(0, x_3)$, or both contained in $(x_4, 1]$, we have that

$$|L_x - L_{x'}| > |x - x'|,$$