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# Automatic differentiation for Fourier series and the radii polynomial approach

Jean-Philippe Lessard [a,*], J.D. Mireles James [b], Julian Ransford [a]

[a] Université Laval, Département de Mathématiques et de Statistique, 1045 avenue de la Médecine, Québec, QC, G1V0A6, Canada

[b] Florida Atlantic University, Department of Mathematical Sciences, Science Building, Room 234, 777 Glades Road, Boca Raton, FL, 33431, USA

## HIGHLIGHTS

- One criticism of the radii polynomial approach is that it appears limited to problems with polynomial nonlinearities.
- Up to now, this approach has been applied only to problems with quadratic and cubic nonlinearities.
- The present work puts to rest this criticism and illustrates the wider applicability of these methods.
- We combine ideas from the theories of automatic differentiation, Fourier series and rigorous numerics.
- We prove the existence of several classical Lyapunov orbits in the PCRTBP.

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ABSTRACT

In this work we develop a computer-assisted technique for proving existence of periodic solutions of nonlinear differential equations with non-polynomial nonlinearities. We exploit ideas from the theory of automatic differentiation in order to formulate an augmented polynomial system. We compute a numerical Fourier expansion of the periodic orbit for the augmented system, and prove the existence of a true solution nearby using an a-posteriori validation scheme (the radii polynomial approach). The problems considered here are given in terms of locally analytic vector fields (i.e. the field is analytic in a neighborhood of the periodic orbit) hence the computer-assisted proofs are formulated in a Banach space of sequences satisfying a geometric decay condition. In order to illustrate the use and utility of these ideas we implement a number of computer-assisted existence proofs for periodic orbits of the Planar Circular Restricted Three-Body Problem (PCRTBP).



## 1. Introduction

Questions concerning the existence of periodic motions are at the heart of the qualitative theory of nonlinear differential equations. When a system is neither a perturbation nor undergoing a Hopf bifurcation, these questions have a global flavor and may be difficult to resolve via classical pen and paper arguments. Numerical simulations provide insight, and in particular may suggest the existence of periodic orbits. In recent years considerable effort has gone into the development of mathematically rigorous a-posteriori methods which close the gap between numerical experiment and mathematical proof.

The present work develops a computer-assisted argument for studying periodic solutions of differential equations with non-polynomial nonlinearities given by elementary functions. Our method consists of formulating a certain fixed point problem on a Banach space of Fourier coefficients, and enables us to obtain quantitative information about analytic properties of the solution. More precisely we study analytic differential equations and obtain analytic parameterizations of periodic orbits in terms of Fourier series, bounds on the decay rate of the coefficients in the tail, lower bounds on the size of the domain of analyticity, and bounds on derivatives of the solution.

In order to implement our computer-assisted argument we borrow an idea from the theory of automatic differentiation. The point is that we must efficiently compose an unknown Fourier series with a non-polynomial vector field. We use "automatic differentiation for Fourier series" to transform the given problem into a polynomial problem in a larger number of variables. The transformed problem is amenable to existing methods of

* Corresponding author.
*E-mail addresses:* jean-philippe.lessard@mat.ulaval.ca (J.-P. Lessard), jmirelesjames@fau.edu (J.D. Mireles James), julian.ransford.1@ulaval.ca (J. Ransford).

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computer-assisted analysis, and we employ the techniques discussed in [1] to complete the argument.

In Section 2.6 we illustrate the utility of the proposed method and give a number of validated periodic orbits for a classical problem of celestial mechanics (namely the planar circular restricted three body problem—PCRTBP). For this problem we apply the automatic differentiation argument in order to obtain the augmented system of polynomial equations, derive the error estimates necessary to apply the method of radii polynomials, and implement the a-posteriori validation. We wish to stress that the orbits, and others similar, to those studied in Section 2.6 have been shown to exist by earlier authors using other methods (see also Section 1.3 below). *The novelty of the present work is a new approach to computer-assisted proofs exploiting automatic differentiation for Fourier analysis of non-polynomial problems*. We choose to work with the PCRTBP because it is a widely studied non-polynomial problem which appears in many serious applications, including a number of computer assisted studies. Before concluding the present introductory discussion, we state Theorem 1, which provides some insight into the nature of our results.

**Theorem 1.** *Let $x(t), y(t)$ be the trigonometric polynomials*

$$x(t) = a_0 + 2\sum_{k=1}^{23} a_k \cos(k\omega t) \quad \text{and} \quad y(t) = -2\sum_{k=1}^{23} b_k \sin(k\omega t),$$

*with $a_k$, $b_k$ the numbers given in Table 1, and $\omega = 1.0102$. Let $\gamma(t) = [x(t), y(t)]$ and $T^* = 2\pi/\omega$. Then there is a real analytic function $\gamma^* \colon [0, T^*] \to \mathbb{R}^2$ denoted component-wise by $\gamma^* = [\gamma_1^*, \gamma_2^*]$ such that*

1. *$\gamma^*$ is a $T^*$-periodic solution of the Planar Circular Restricted Three-Body Problem (PCRTBP) given in (2.1) with mass parameter $\mu = 0.0123$. (The PCRTBP is discussed in Section 2).*
2. *$\gamma^*$ is a symmetric solution in the sense that $\gamma_1^*(t)$ is given by a cosine series and $\gamma_2^*(t)$ is given by a sine series.*
3. *$\gamma^*$ is $C^0$ close to $\gamma$. More precisely, for $r = 9.93 \times 10^{-11}$,*
$$\sup_{t\in[0,T^*]} \left|\gamma_1^*(t) - x(t)\right| \le r \quad \text{and} \quad \sup_{t\in[0,T^*]} \left|\gamma_2^*(t) - y(t)\right| \le r.$$
4. *The function $\gamma^*$ can be extended to a $T$-periodic analytic function on a complex strip having width at least*
$$\ln(1.14)/\omega \approx 0.1297.$$
5. *The decay rates of the Fourier coefficients $a_k^*$ of $\gamma_1^*$ and $b_k^*$ of $\gamma_2^*$ satisfy the bounds*
$$|a_k^*|, |b_k^*| \le \frac{7.37 \times 10^{-10}}{1.14^k}, \quad \text{for } k \ge 24.$$

*The orbit itself is illustrated in Fig. 4.*

The remainder of the paper is organized as follows: The remaining sections of the introduction (Sections 1.1–1.3) are devoted to some brief discussion of the literature pertaining to respectively the method of radii polynomials, the technique known as automatic differentiation, and distinctions between the geometric/phase space versus the functional analytic approach to computer-assisted proof in nonlinear analysis. These sections motivate the work to follow. In Sections 1.4 and 1.5, we review the basic notions of Automatic Differentiation for Taylor series and describe the situation for Fourier series. In Section 2, we begin discussing the Planar Circular Restricted Three-Body Problem, the main example of the paper. We review the equations of motions and in Section 2.1, we illustrate the Automatic Differentiation scheme for the problem. Then, in Sections 2.2 and 2.3 we develop the appropriate Banach Spaces and the associated $F(x) = 0$ problem. In Sections 2.4 and 2.5, we develop the Newton-like operator and the radii polynomials for the problem. Finally in Section 2.6 we present the results of a number of computer-assisted proofs.

**Table 1**
The Fourier coefficients of $(x(t), y(t))$ of the inner most green periodic orbits on the left in Fig. 4. The frequency of the orbit is $\omega = 1.0102$. There are 24 Fourier coefficients per component. We could prove the existence of the orbit with $\nu = 1.14$ and $r = 7.81 \times 10^{-10}$. We also proved the existence with $\nu = 1.09$ and $r = 1.1 \times 10^{-10}$.

| $k$ | $a_k$ | $b_k$ |
|---|---|---|
| 0 | $-9.768220550865100 \times 10^{-1}$ | – |
| 1 | $-1.206409736493824 \times 10^{-1}$ | $-2.339364883946253 \times 10^{-1}$ |
| 2 | $-1.347990876182309 \times 10^{-2}$ | $6.397869958624213 \times 10^{-3}$ |
| 3 | $2.352585563377883 \times 10^{-3}$ | $-1.790561590462826 \times 10^{-3}$ |
| 4 | $-5.122863209781185 \times 10^{-4}$ | $4.373987650320329 \times 10^{-4}$ |
| 5 | $1.219044853784686 \times 10^{-4}$ | $-1.093231659117833 \times 10^{-4}$ |
| 6 | $-3.055685576355795 \times 10^{-5}$ | $2.814155571301070 \times 10^{-5}$ |
| 7 | $7.934843979226718 \times 10^{-6}$ | $-7.428978433135244 \times 10^{-6}$ |
| 8 | $-2.114596180594020 \times 10^{-6}$ | $2.001931208169388 \times 10^{-6}$ |
| 9 | $5.748968790488728 \times 10^{-7}$ | $-5.486299609956681 \times 10^{-7}$ |
| 10 | $-1.588047255749821 \times 10^{-7}$ | $1.524601425602110 \times 10^{-7}$ |
| 11 | $4.444105347947109 \times 10^{-8}$ | $-4.286470381069696 \times 10^{-8}$ |
| 12 | $-1.257223241334567 \times 10^{-8}$ | $1.217145490467260 \times 10^{-8}$ |
| 13 | $3.589443646509739 \times 10^{-9}$ | $-3.485577927995539 \times 10^{-9}$ |
| 14 | $-1.032912424206540 \times 10^{-9}$ | $1.005555404303114 \times 10^{-9}$ |
| 15 | $2.992744553664615 \times 10^{-10}$ | $-2.919684209251029 \times 10^{-10}$ |
| 16 | $-8.723250537889003 \times 10^{-11}$ | $8.525779373911926 \times 10^{-11}$ |
| 17 | $2.556153430692412 \times 10^{-11}$ | $-2.502217287937952 \times 10^{-11}$ |
| 18 | $-7.525630237426317 \times 10^{-12}$ | $7.376943496986842 \times 10^{-12}$ |
| 19 | $2.225016823097366 \times 10^{-12}$ | $-2.183690063445679 \times 10^{-12}$ |
| 20 | $-6.603543558111446 \times 10^{-13}$ | $6.487830301502292 \times 10^{-13}$ |
| 21 | $1.966611110890260 \times 10^{-13}$ | $-1.933996961613629 \times 10^{-13}$ |
| 22 | $-5.875220903859556 \times 10^{-14}$ | $5.782732245351016 \times 10^{-14}$ |
| 23 | $1.760271955315164 \times 10^{-14}$ | $-1.733885618947898 \times 10^{-14}$ |

### *1.1. Computer-assisted proofs and the radii polynomial approach*

The *radii polynomial approach* refers to a tool kit for *a posteriori* computer-assisted verification of the existence of a zero of a nonlinear operator equation

$$F(x) = 0 \tag{1.1}$$

defined on an infinite-dimensional Banach space. The solution $x$ may represent an invariant set of a dynamical system like a steady state, a periodic orbit, a connecting orbit, a stable manifold, etc. It could also be a minimizer of an action functional, an eigenpair of an eigenvalue problem or a solution to a boundary value problem. The radii polynomial approach consists of taking a finite-dimensional projection of (1.1), computing an approximate solution $\bar{x}$ (e.g. using Newton's method), constructing an approximate inverse $A$ of $DF(\bar{x})$, and then proving the existence of a fixed point for the Newton-like operator

$$T(x) \stackrel{\text{def}}{=} x - AF(x) \tag{1.2}$$

by applying the Contraction Mapping Theorem (CMT) on closed balls about $\bar{x}$. The hypotheses of the CMT are rigorously verified by deriving a system of polynomial equations (the radii polynomials) whose coefficients carry the relevant information about the nonlinear mapping (1.2), the topology of the solution space, the given numerical approximate solution $\bar{x}$, and the choice of approximate inverse $A$ for the derivative of the mapping $F$. The question "Is $T$ a contraction on some neighborhood of $\bar{x}$ ?" is reduced to a question about the sign of some polynomials. By studying the maximum interval on which these polynomials are negative we obtain bounds on the size of both the smallest and largest neighborhoods on which $T$ is a contraction. These give rise respectively to the best computer-assisted error and isolation bounds which can be obtained relative to the choices defining $T$. We note that while error bounds are of most interest, the isolation bounds are also useful in some validated continuation arguments [2,3].

Before proceeding any further, we hasten to mention the existence of several other computer-assisted methods based on the