GENERALIZED KOTANI'S TRICK FOR UNITARY OPERATORS

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(*Received April 28, 2016 – Revised October 17, 2016*)

We present generalizations of Kotani's trick for unitary operators, including a Hausdorff continuous version adapted from recent results in the self-adjoint case.

Keywords: rank one perturbations, unitary operators, Kotani's trick, Hausdorff continuity.

1. Introduction and results

Rank-one perturbations of a unitary operator U is a subject of physical and mathematical interest. Denote by $|\phi\rangle\langle\phi|$ the projection operator on the subspace spanned by the vector ϕ in a (separable) Hilbert space; for $\lambda \in \mathbb{R}$ such perturbations have the form

$$U_\lambda := U \exp(i\lambda|\phi\rangle\langle\phi|) = U\left[1 + (e^{i\lambda} - 1)|\phi\rangle\langle\phi|\right]. \tag{1.1}$$

For simplicity, we normalize ϕ.

From a physical viewpoint, if H_0 represents an unperturbed self-adjoint operator, U_λ corresponds to the Floquet operator of the periodically kicked Hamiltonian formally given by [3, 6–8]

$$H(t) = H_0 + \lambda\,|\phi\rangle\langle\phi| \sum_{n\in\mathbb{Z}} \delta(t-n),$$

so that in (1.1) we identify $U = e^{-iH_0}$ and λ is the intensity of the kicks. Note that $\lambda \mapsto U_\lambda$ is 2π-periodic, and so one may restrict the considerations to $\lambda \in [0, 2\pi]$. See [22] for the case of periodically rank-N kicked Hamiltonian and [1] for the question of cyclicity and Alekssandrov–Clark theory related to unitary rank-one perturbations.

There is large literature on rank-one perturbations of self-adjoint operators (see the review [26] that is still a key reference), and a useful tool in this setting is the so-called Kotani's trick [18–20], a result on spectral averaging with many applications (see [5, 10–12, 14, 29] and references therein). There is more restricted literature in the case of unitary operators.

The first version of this trick in the unitary situation appeared in [6] (see Lemma 5 there), and it says that a particular choice of an absolutely continuous (with respect to Lebesgue) measure ρ in (1.3) results in Ω_ρ equivalent to Lebesgue measure. It was important in the derivation of a unitary version [6] of the so-called Simon–Wolff criterion [29]. This set of results was applied to different (although related) contexts involving unitary operators: Anderson localization in the random unitary framework [15], studies of the Chalker–Coddington model [2], Floquet operators with pure point spectrum and energy instability [9], singular continuous spectra [3, 4] for some Floquet operators, onset of quantum chaos [23], adaptation of the fractional moment method to random unitary operators [17]. Such applications, besides the interest in spectral averages in the unitary setting in their own right, motivated us to investigate possible unitary versions of results by Marx [20] on rank-one perturbations of self-adjoint operators.

The work [20] has considered the spectral averaging through a more general measure η by extending the case where η is the Lebesgue measure, and examined how continuity properties of η are inherited by the final result; in particular when Hausdorff dimensional properties are considered. Here we present extensions of such results to the unitary setting (1). Although the general ideas are borrowed from [20], the technicalities in the unitary case are more involved and require precise choice of working functions, as discussed in the proofs ahead.

Our first results (i.e. Theorems 1 and 2) compose a more detailed version of spectral averages when ρ below is Lebesgue measure; denote by $\ell(\cdot)$ the Lebesgue measure restricted to the Borel sets of $[0, 2\pi]$ (with the ends identified), and by ω and ω_λ the spectral measures of U and U_λ, respectively, both with respect to ϕ.

THEOREM 1. *Consider the measure Γ defined on Borel sets $B \subset [0, 2\pi]$ by the spectral averaging with respect to Lebesgue measure*

$$\Gamma(B) := \int_0^{2\pi} \omega_\lambda(B) d\lambda. \tag{1.2}$$

Then, $\Gamma = \ell$.

THEOREM 2. *Fix a finite measure ρ on the Borel sets B of $[0, 2\pi]$ and define the spectral averaging measure Ω_ρ by*

$$\Omega_\rho(B) := \int_0^{2\pi} \omega_\lambda(B) d\rho(\lambda). \tag{1.3}$$

Then if $\rho \ll \ell$ one also has $\Omega_\rho \ll \ell$.

For $0 \le \alpha \le 1$, denote by h^α the α-Hausdorff measure restricted to $[0, 2\pi]$ [13, 21, 24]. Recall that a Borel measure μ is *α-Hausdorff continuous* (α-Hc) if $\mu(B) = 0$ for all sets with $h^\alpha(B) = 0$, and it is α-Hausdorff singular (α-Hs) if there is a Borel set B with $h^\alpha(B) = 0$ and $\mu(B^c) = 0$. Also, $h^1 = \ell$ on measurable subsets of the real line, so that a measure μ is 1-Hc if and only if $\mu \ll \ell$.