ORTHOMODULAR LATTICE IN LORENTZIAN GLOBALLY HYPERBOLIC SPACE-TIME

WOJCIECH CEGŁA[1*], JAN FLOREK[2] and BERNARD JANCEWICZ[1]

[1]Institute of Theoretical Physics, University of Wrocław,
pl. Maksa Borna 9, PL-50-204 Wrocław, Poland
(e-mails: wojciech.cegla@ift.uni.wroc.pl, bernard.jancewicz@ift.uni.wroc.pl)
[2]Institute of Mathematics, University of Economics,
ul. Komandorska 118/120, PL-53-345 Wrocław, Poland
(e-mail: jan.florek@pwr.edu.pl)

(*Received June 14, 2016 – Revised August 31, 2016*)

An orthomodular lattice without covering law is considered in globally hyperbolic space-time where orthogonality is generated by the chronological relation. In this lattice, the least upper bound and orthocomplementation cannot be interpreted as the disjunction and negation of classical logic. Two-dimensional pictures are presented, demonstrating nonclassical character of the lattice. M_3-N_5 theorem is used to consider nonmodularity. A comparison of causal logic and quantum logic is discussed.

Keywords: causal structure, quantum logic, orthogonality, orthomodularity, covering law, M_3-N_5 theorem.

1. Introduction

In the previous paper [1] nonmodular lattices were considered in the Lorentzian space-time where notions of chronology and causality were defined. In the present paper we consider a lattice structure approach in relativity theory by constructing an orthomodular lattice in globally hyperbolic space-time. The orthomodularity is a key ingredient in the standard quantum logic [2] and its appearance in relativity seems to be unexpected and could suggest a kind of connection between general relativity and quantum theory.

In Minkowski space-time $M = \mathbb{R} \times \mathbb{R}^3$ where $\mathbb{R}$ represents the time axis and $\mathbb{R}^3$ the physical 3-space, a light cone is given by a finite speed of light. Any kind of dynamics of a massive particle is possible inside the light cone only and each movement could be considered as an admissible time-like curve. At the same time the light cone generates an orthogonality relation in M in the following way: for $x, y \in M$, $x \neq y$ one puts $x \perp y$ if and only if $x - y$ is space-like or light-like, which is related to Einstein's causality principle. Non-signalling by a time-like curve

[*]Supported by grant no. 1009/S/IFT/15 of Wrocław University.

means orthogonality. The orthogonality relation is used to build an orthomodular lattice in the space-time.

One can generalize this investigation from the Minkowski space-time to a globally hyperbolic space-time where x and y are orthogonal if they are not connected by a chronological path. Having the orthogonality relation, we define the family of double orthoclosed sets

$$\mathcal{L}(M, \perp) := \{A \subset M;\ A = A^{\perp\perp}\}.$$

In Section 2 we show that although $\mathcal{L}(M, \perp)$ arose from a part of classical (nonquantum) physics, it is not Boolean but only an orthomodular lattice as the standard quantum logic is. The starting point for our investigation is a properly chosen family G of causal paths which has a connection to the causal structure of globally hyperbolic space-time [3].

There is a short list of references concerning this subject. Items [4–6] are first papers combining causal structure and corresponding lattice structure in the case of Galilean and Minkowski space-time. The main intriguing result, not widely known, i.e. orthomodularity of the so-called causal logic $\mathcal{L}(M, \perp)$ of Minkowski space-time is contained in [5]. It was a seminal paper for other authors, see [7, 8]. This became interesting for the research in the domain of logic [9]. Orthomodular lattices generated by graphs of continuous functions were constructed in [10–12]. This was a starting point for investigation of orthomodular lattice in the ordered vector spaces [13, 14]. A connection with Petri's nets was mentioned recently [15].

In Section 3 we show nonclassical character of the causal logic. We illustrate this in two-dimensional Minkowski space-time. Using M_3-N_5 theorem we discuss nonmodularity of the causal logic. Finally, in Section 4 we compare quantum and causal logics.

2. Orthomodular lattice in globally hyperbolic space-time

At the beginning of the present section we consider a causal structure and a lattice structure with the orthogonality relation generated by causality. Later on, we exploit the results from [10, 11] in general model of space-time $M = \mathbb{R} \times X$ which is a topological product of the real line $\mathbb{R}$, which represents time, and an arbitrary T_1 topological space X.

Causal structure. In the space-time M, we introduce a distinguished family G of subsets f of M covering M, i.e. $M = \bigcup_{f \in G} f$. The covering family G is called *causal structure* and the elements f of G—*causal paths*. Physically one can interpret G as the set of all possible signals in the space-time M. As the family G we consider the set of graphs of continuous functions $f : \mathbb{R} \to X$, identifying the function and its graph. The causal structure G in the space-time $M = \mathbb{R} \times X$ generates an ordered space $(M, \leq)_G$ with antisymmetric relation $\leq$ as follows

$$x \leq y \quad \text{if there exists} \quad f \in G \quad \text{such that} \quad \{x, y\} \subseteq f \quad \text{and} \quad p(x) \leq p(y),$$

where $p(x)$ is the canonical projection of $\mathbb{R} \times X$ onto $\mathbb{R}$.