THEORETICAL FOUNDATIONS OF INCORPORATING LOCAL BOUNDARY CONDITIONS INTO NONLOCAL PROBLEMS

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We study nonlocal equations from the area of peridynamics on bounded domains. We present four main results. In our recent paper, we have discovered that, on $\mathbb{R}$, the governing operator in peridynamics, which involves a convolution, is a bounded function of the classical (local) governing operator. Building on this, as main result 1, we construct an abstract convolution operator on bounded domains which is a generalization of the standard convolution based on

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integrals. The abstract convolution operator is a function of the classical operator, defined by a Hilbert basis available due to the purely discrete spectrum of the latter. As governing operator of the nonlocal equation we use a function of the classical operator, this allows us to incorporate local boundary conditions into nonlocal theories. As main result 2, we prove that the solution operator can be uniquely decomposed into a Hilbert–Schmidt operator and a multiple of the identity operator. As main result 3, we prove that Hilbert–Schmidt operators provide a smoothing of the input data in the sense a square integrable function is mapped into a function that is smooth up to boundary of the domain. As main result 4, for the homogeneous nonlocal wave equation, we prove that continuity is preserved by time evolution. Namely, the solution is discontinuous if and only if the initial data is discontinuous. As a consequence, discontinuities remain stationary.

Keywords: nonlocal wave equation, nonlocal operator, peridynamics, boundary condition, Hilbert–Schmidt operator, operator theory.

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1. Introduction

There are indications that a development of nonlocal theories is necessary for description of certain natural phenomena. It is a part of the folklore in physics that the point particle model, which is the root for *locality* in physics, is the cause of unphysical singular behaviour in the description of phenomena. On the other hand, all fundamental theories of physics are local. There are many more alternatives for formulating nonlocal theories compared to local ones. Therefore, the task of formulating a viable nonlocal theory consistent with experiment seems much harder than that of a local one. In any case, for such formulation, an understanding of the capacities of nonlocal theories appears inevitable.

Considering wave phenomena only partially successfully described by a classical wave equation, it seems reasonable to expect that a more successful model can be obtained by employing the functional calculus of self-adjoint operators, i.e. by replacing the classical governing operator A by a suitable function $f(A)$. We call f the *regulating function*. By choosing different regulating functions, we can define governing operators tailored to the needs of the underlying application. Since classical boundary conditions (BC) is an integral part of the classical operator, these BC are automatically inherited by $f(A)$. In this way, we vision to model wave phenomena by using appropriate $f(A)$ and, as a consequence, need to study the effect of $f(A)$ on the solutions. One advantage of our approach is that every symmetry that commutes with A also commutes with $f(A)$. As a result, required invariance with respect to classical symmetries such as translation, rotation and so forth is preserved. The choice of regulating functions appropriate for the physical situation at hand is an object for future research.

We are interested in studying instances of successful modeling by nonlocal theories of phenomena that cannot be captured by local theories. There are noteworthy developments in the area of nonlocal modeling. For instance, crack propagation [1] and viscoelastic damping [2] are modeled by peridynamics (PD) and fractional derivatives, respectively, both of which are nonlocal. PD is a nonlocal extension of continuum mechanics developed by Silling [1], is capable of quantitatively predicting the dynamics of propagating cracks, including bifurcation. Its effectiveness has been