DUALITY FOR GRADED MANIFOLDS*

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We study the notion of duality in the context of graded manifolds. For graded bundles, somehow like in the case of Gelfand representation and the duality: points vs. functions, we obtain natural dual objects which belong to a different category than the initial ones, namely graded polynomial (co)algebra bundles and free graded Weil (co)algebra bundles. Our results are then applied to obtain elegant characterizations of double vector bundles and graded bundles of degree 2. All these results have their supergeometric counterparts. For instance, we give a simple proof of a nice characterisation of N-manifolds of degree 2, announced in the literature.

Keywords: graded bundle, duality, homogeneity structure, N-manifold, Zakrzewski morphism.

1. Introduction

1.1. Physical motivations

In this work we study the concept of duality for certain classes of graded manifolds. Since we do not give any explicit physical applications, it is worth discussing the physical motivations behind this research. Graded bundles, the main topic of this study, were introduced in [1] as natural generalizations of vector bundles. The most important examples of graded bundles are the bundles of higher velocities (higher tangent bundles), hence graded bundles form a natural geometric framework for geometric mechanics (see [2] for a direct application in higher-order Lagrangian mechanics). What is more, double vector bundles and graded supermanifolds (N-manifolds), also discussed in our paper, are now widely recognized as laying at the foundations of geometric mechanics. Note also that the

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duality of (double) vector bundles plays an important role in the construction of the so-called Tulczyjew triple [3], which encodes the construction of the Lagrangian and the Hamiltonian formalism in mechanics.

For the above reasons we believe that the current study, covering, in particular, a question "what is the dual object to the bundles of higher velocities (higher tangent bundles)?", may contribute to a better understanding of the geometric foundations of mechanics. Below, starting from a careful discussion of the duality for vector spaces, we motivate our ideas about the concept of duality for graded bundles.

1.2. Duality for vector spaces

In linear algebra the notion of duality can be understood as a functor between the category $\mathcal{V}ect$ of finite-dimensional vector spaces with linear maps and the opposite category $\mathcal{V}ect^{\mathrm{op}}$.

Recall that given a real vector space V, its dual is defined as the set of all linear maps from V to the model space $\mathbb{R}$,

$$V^* := \mathrm{Hom}_{\mathrm{lin}}(V, \mathbb{R}). \tag{1}$$

The resulting set V^* possesses a natural structure of a vector space induced by the linear structure on $\mathbb{R}$. Let now $\psi : V \to W$ be a linear map. Its dual

$$\psi^* : W^* \longrightarrow V^*,$$

defined by the formula $\psi^*(h) := h \circ \psi$, where $h : W \to \mathbb{R}$ is an element of W^*, is again a liner map. In this way one constructs a functor

$$\mathcal{V}ect \xrightarrow{\ *\ } \mathcal{V}ect^{\mathrm{op}},$$

which is, in fact, an equivalence of categories of finite-dimensional vector spaces. This follows from the well-known fact that the space $(V^*)^*$ is canonically isomorphic to V (every linear map from V^* to $\mathbb{R}$ is an evaluation).

1.3. Duality for vector bundles

The above construction can be straightforwardly extended to the *category* of vector bundles and vector bundle morphisms $\mathcal{VB}$. The fiber-wise application of the functor $*$ produces a natural equivalence of categories

$$\mathcal{VB} \underset{\star}{\overset{*}{\rightleftarrows}} \mathcal{VB}^*.$$

The inverse functor $\star$ is also obtained by the fiber-wise application of the vector space duality. Note that functor $*$ is not valued in the category of vector bundles $\mathcal{VB}$, but rather in $\mathcal{VB}^*$, which symbol denotes the category of vector bundles with morphisms in the sense of Zakrzewski [4], known in the literature also as *vector bundle morphisms of the second kind* [5], *star bundle morphisms* [6], or *comorphisms* [7]. More precisely one has definition,