A UNIFIED STUDY OF ORTHOGONAL POLYNOMIALS VIA LIE ALGEBRA

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In this paper, we discuss some operators defined on Lie algebras for the purpose of deriving properties of some special functions. The method developed in this paper can also be used to study some other special functions of mathematical physics. We have established a general theorem concerning eigenvectors for the product of two operators defined on a Lie algebra of endomorphisms of a vector space. Further, using this result, we have obtained differential recurrence relations and differential equations for the extended Jacobi polynomials and the Gegenbauer polynomials. Results of many researchers; see for example Radulescu (1991), Mandal (1991), Pathan and Khan (2003), Humi, and the references therein, follow as special cases of our results.

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1. Introduction

The theory of generalized special functions has witnessed a rather significant evolution during the last years. The most widely used orthogonal polynomials are the classical orthogonal polynomials, consisting of the Hermite polynomials, the Laguerre polynomials, the Jacobi polynomials together with their special cases, see, e.g., Dattoli et al. [7–10]). One of important methods for studying special functions via their recurrence relations and differential equations lies closely to the standard Lie algebraic techniques. Many important classical differential equations have connection with Lie theory. The interplay between differential equations, special functions and Lie theory plays an important role in mathematical physics. When the Lie algebraic aspects of special functions are considered in the literature, they

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are limited to the Lie algebras generated by the raising, lowering and maintaining operators.

Radulescu [19, 20] discussed some important properties of Hermite and Laguerre polynomials using some operators defined on Lie algebras. Mandal [2] studied some properties of simple Bessel polynomials considered by Krall and Frink [12]. Pathan and Khan [17, 18] extended the Lie algebraic approach discussed by Radulescu [19] and Mandel [2] to derive some properties of generalized Hermite polynomials of two variables (see e.g. Dattoli et al. [4]), generalized Bessel functions of two variables (Dattoli et al. [5, 6]) and two variable Laguerre polynomials (Pathan and Khan [17, 18]). Recently, Humi [16] has shown that in addition to these operators, the dilation and the translation operators can be added to these Lie algebras for some families of factorisable equations using factorization method used in theoretical physics.

Our object in this paper is to obtain a theorem using some operators defined on Lie algebras which generalizes many results of researchers to the families of special functions and orthogonal polynomials which were discussed above. Some additional applications of the action of these operators on the families of special functions are given. In particular, we present examples how the Lie algebraic technique can be used to derive the differential recurrence relations, differential equations and the Rodrigue type formula for the extended Jacobi polynomials and the Gegenbauer polynomials.

The Jacobi polynomials appear naturally as extension of the Legendre and the Gegenbauer polynomials in the context of potential theory and harmonic analysis. The Jacobi polynomials have been used extensively in mathematical analysis and many practical applications. Fujiwara [13] studied the polynomials $F_n^{(\alpha,\beta)}(x;a,b,c)$ which are called the extended Jacobi polynomials defined by the Rodrigue formula

$$F_n^{(\alpha,\beta)}(x;a,b) = \frac{(-1)^n}{n!}\left(\frac{\lambda}{b-a}\right)^n (x-a)^{-\alpha}(b-x)^{-\beta}D^n[(x-a)^{n+\alpha}(b-x)^{n+\beta}], \quad (1)$$

where $D = \frac{d}{dx}$, $\alpha, \beta > -1$.

Thakare [15] showed that

$$\begin{aligned} F_n(\alpha,\beta;x) &= \lambda^n \left(\frac{x-a}{b-a}\right)^n \binom{n+\beta}{n} {}_2F_1\left[\begin{matrix}-n, -n-a \\ 1+\beta\end{matrix}; \frac{x-b}{x-a}\right] \\ &= \lambda^n \left(\frac{x-a}{b-a}\right)^n \binom{n+\beta}{n} \sum_{n=0}^{\infty} \frac{(-n)_k(-n-a)_k}{k!(1+\beta)}\left(\frac{x-b}{x-a}\right)^k, \end{aligned} \quad (2)$$

and also expressed the extended Jacobi polynomials as

$$\begin{aligned} F_n(\alpha,\beta;x) &= \lambda^n \binom{n+\beta}{n} {}_2F_1\left[\begin{matrix}-n, 1+\alpha+\beta+n \\ 1+\beta\end{matrix}; \frac{b-x}{b-a}\right] \\ &= \lambda^n \left(\frac{x-a}{b-a}\right)^n \binom{n+\beta}{n} \sum_{n=0}^{\infty} \frac{(-n)_k(1+\alpha+\beta+n)_k}{k!(1+\beta)}\left(\frac{x-b}{x-a}\right)^k. \end{aligned} \quad (3)$$