STATES AND SYNAPTIC ALGEBRAS

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Different versions of the notion of a state have been formulated for various so-called quantum structures. In this paper, we investigate the interplay among states on synaptic algebras and on its sub-structures. A synaptic algebra is a generalization of the partially ordered Jordan algebra of all bounded self-adjoint operators on a Hilbert space. The paper culminates with a characterization of extremal states on a commutative generalized Hermitian algebra, a special kind of synaptic algebra.

Keywords: synaptic algebra, GH-algebra, Jordan algebra, convex effect algebra, MV-algebra, ℓ-group, order unit normed space, state, extremal state.

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1. Introduction

Synaptic algebras, featured in this paper, incorporate several so-called "quantum structures." Quantum structures were originally understood to be mathematical systems that permit a perspicuous representation for at least one of the key ingredients of quantum-mechanical theory, e.g. states, observables, symmetries, properties, and experimentally testable propositions [11, 13, 40]. In spite of the adjective 'quantum,' a variety of mathematical structures arising in classical physics, computer science, psychology, neuroscience, fuzzy logic, fuzzy set theory, and automata theory are now regarded as quantum structures.

As per the title of this paper, which is intended to complement the authoritative articles [9, 10] by A. Dvurečenskij, we shall study the interplay among states on a synaptic algebra and on some of its sub-structures that qualify as quantum structures. A state assigns to a real observable (a physical quantity) the expected value of the observable when measured in that state. Also, a state assigns to a testable 2-valued proposition the probability that the proposition will test 'true' in that state.

The adjective ‘synaptic’ is derived from the Greek word ‘sunaptein,’ meaning to join together; indeed synaptic algebras unite the notions of an order-unit normed space [1, p. 69], a special Jordan algebra [34], a convex effect algebra [3, 6], and an orthomodular lattice [4, 33].

Since virtually every quantum structures, including synaptic algebras, are partially ordered sets (posets for short), we review some of the basic definitions and facts concerning posets in Section 2. Also since every synaptic algebra A is an extension of a so-called convex effect algebra E, and the states on A are in affine bijective correspondence with the states on E, we offer a brief review of effect algebras and states thereon in Section 3. Moreover, every synaptic algebra is an order-unit normed linear space, a structure that we review in Section 4, where states on an order-unit normed space are defined and some of their properties are recalled.

Section 5 is devoted to a brief account of some of the basic properties of synaptic algebras (SAs for short) and to a special case thereof called generalized Hermitian (GH-) algebras. States on an SA are defined just as they are for any order-unit normed space. In Section 6, we consider commutative SAs and their functional representations. In Section 7 we characterize extremal states on commutative GH-algebras.

In what follows, the notation := means ‘equals by definition,’ the phrase ‘if and only if’ is abbreviated as ‘iff,’ $\mathbb{N} := \{1, 2, 3, ...\}$ is the system of natural numbers, the ordered field of real numbers is denoted by $\mathbb{R}$, and $\mathbb{R}^+ := \{\alpha \in \mathbb{R} : 0 \leq \alpha\}$.

2. Partially ordered sets

A binary relation $\leq$ defined on a nonempty set $\mathcal{P}$ is a *partial order relation* iff for all $a, b, c \in \mathcal{P}$, (1) $a \leq a$, (2) $a \leq b$ and $b \leq a \Rightarrow a = b$, and (3) $a \leq b$ and $b \leq c \Rightarrow a \leq c$. A *partially ordered set* (*poset* for short) is a nonempty set $\mathcal{P}$ equipped with a distinguished partial order relation $\leq$.

Suppose that $\mathcal{P}$ is a poset with partial order relation $\leq$ and let $a, b \in \mathcal{P}$. We write $b \geq a$ iff $a \leq b$, and the notation $a < b$ (or $b > a$) means that $a \leq b$ but $a \neq b$. Let $\mathcal{Q} \subseteq \mathcal{P}$. We say that a is a *lower bound*, (b is an *upper bound*), for $\mathcal{Q}$ iff $a \leq q$, ($q \leq b$), for all $q \in \mathcal{Q}$. Also, a is the *least* or the *minimum*, (b is the *greatest* or the *maximum*) element of $\mathcal{Q}$ iff a is a lower bound for $\mathcal{Q}$ and $a \in \mathcal{Q}$, (b is an upper bound for $\mathcal{Q}$ and $b \in \mathcal{Q}$). The notation $a = \bigwedge \mathcal{Q}$, ($b = \bigvee \mathcal{Q}$), means that a is the greatest lower bound, (b is the least upper bound), of $\mathcal{Q}$. The greatest lower bound $a = \bigwedge \mathcal{Q}$, (the least upper bound $b = \bigvee \mathcal{Q}$), if it exists, is also called the *infimum*, (the *supremum*), of $\mathcal{Q}$ in $\mathcal{P}$.

If it exists, the greatest lower bound, (least upper bound), of the set $\{a, b\} \subseteq \mathcal{P}$ is written as $a \wedge b$, (as $a \vee b$) and is often referred to as the *meet* (the *join*) of a and b. If it is necessary to make clear that an existing meet $a \wedge b$ or join $a \vee b$ is calculated in $\mathcal{P}$, it may be written as $a \wedge_{\mathcal{P}} b$ or $a \vee_{\mathcal{P}} b$. The poset $\mathcal{P}$ is said to be *lattice ordered*, or simply a *lattice* iff every pair of elements in $\mathcal{P}$ has a meet and a join in $\mathcal{P}$. By definition, a lattice $\mathcal{P}$ is *distributive* iff, for all $a, b, c \in \mathcal{P}$, $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$, or equivalently $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$.