CLIFFORD TORI AND UNBIASED VECTORS

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The existence problem for mutually unbiased bases is an unsolved problem in quantum information theory. A related question is whether every pair of bases admits vectors that are unbiased to both. Mathematically this translates to the question whether two Lagrangian Clifford tori intersect, and a body of results exists concerning it. These results are however rather weak from the point of view of the first problem. We make a detailed study of how the intersections behave in the simplest nontrivial case, that of complex projective 2-space (the qutrit), for which the set of pairs of Clifford tori can be usefully parametrized by the unistochastic subset of Birkhoff's polytope. Pairs that do not intersect transversally are located. Some calculations in higher dimensions are included to see which results are special to the qutrit.

Keywords: mutually unbiased bases, Lagrangian submanifolds, Birkhoff's polytope.

1. Introduction

The existence problem for Mutually Unbiased Bases (MUB) is easy to state, arises naturally when asking questions about the foundations of quantum mechanics, and may have various practical implications [1]. After several decades of work by quantum physicists, it remains open. From a mathematical point of view it is not clear how to address this problem. It just might be bound up with unsolved questions in discrete mathematics such as the existence problem for finite projective planes [2], as has been discussed in the literature [3–5]. As noticed recently it may also be related to questions in Lagrangian intersection theory, an at first sight unrelated area of mathematics with roots in very different soil [6].

On the physics side the set of all vectors unbiased with respect to a given basis can be regarded as 'maximally quantum' states, as seen by an observer having access to only one von Neumann measurement. In mathematics this set of vectors is known as a Clifford torus, and forms a Lagrangian submanifold of complex projective space. Thus, listing the set of all vectors unbiased to two different bases is equivalent to the problem of enumerating the intersections between two Clifford tori. Provided the intersections are transversal (they may not be) a powerful mathematical theorem guarantees that at least 2^{N-1} distinct intersections occur when the Hilbert space has N dimensions [7, 8]. However, to be of use in the MUB existence problem these intersections have to occur in a very special constellation.

The case $N = 2$ is trivial. In this case a Clifford torus is a great circle on the Bloch sphere. Two great circles always intersect in two antipodal points, and these points represent an orthonormal basis unbiased to the two bases one started out with. The purpose of this paper is to investigate the simplest nontrivial case, $N = 3$, in full detail. Using a natural parametrization of the set of all pairs of Clifford tori we find that the number of intersections is 4 or 6, with 3 or 5 intersections occurring in exceptional nontransversal cases.

Section 2 briefly reviews the MUB existence problem and the subject of biunimodular vectors. Section 3 explains why questions about intersecting Lagrangian tori are relevant to it, and recalls some facts about intersecting submanifolds. Section 4 parametrizes the set of Clifford tori by means of the unistochastic subset of Birkhoff's polytope. A detailed description of the $N = 3$ case is given since it has independent interest. Sections 5 and 6 present our main result, a description of how the regions with four and six intersection points sit inside the parameter space. We pay special attention to sets of measure zero where the transversality condition fails. Most of the story is based on Mathematica calculations, with some analytical support. We believe that the resulting picture is basically complete—when $N = 3$. The question whether it has any implications for the MUB existence problem is deferred to the concluding Section 7. Some analytical calculations for higher dimensions are given in two appendices, to see to what extent our results are special to the $N = 3$ case.

2. The MUB existence problem and biunimodular vectors

In quantum mechanics an orthonormal basis is needed to specify a von Neumann measurement. Given the Hilbert space $\mathbb{C}^N$, two orthonormal bases $\{|e_i\rangle\}_{i=1}^N$ and $\{|f_i\rangle\}_{i=1}^N$ are said to be *unbiased* if all the numbers $|\langle e_i|f_j\rangle|^2$ are the same. Thus the transition probabilities are

$$p(i|j) = |\langle e_i|f_j\rangle|^2 = \frac{1}{N} \tag{1}$$

for all i, j. Pairs of bases related in this way have practical applications to quantum cryptography, and the concept is closely related to Bohr's notion of complementarity [9], supposedly central to the philosophy of quantum mechanics. One obtains an example of an unbiased pair for any dimension N by letting the first basis be the computational basis, and then acting on its vectors by means of the Fourier matrix F, with matrix elements

$$F_{jk} = \frac{1}{\sqrt{N}} e^{\frac{2\pi i}{N} jk}, \qquad 0 \leq j, k \leq N - 1. \tag{2}$$

The resulting basis is called the Fourier basis. Its vectors are made up of the columns of the Fourier matrix.

In the *MUB existence problem* one goes on to ask how many mutually unbiased bases there can be, given N. An answer would have implications ranging from