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ON THE DIMENSIONS OF OSCILLATOR-LIKE ALGEBRAS INDUCED BY ORTHOGONAL POLYNOMIALS: NONSYMMETRIC CASE

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There is a generalized oscillator-like algebra associated with every class of orthogonal polynomials, on the real line, satisfying a four-term nonsymmetric recurrence relation. This note presents necessary and sufficient conditions, on the coefficients of the recurrence relation, for such algebras to be of finite dimension. As examples, we discuss the dimensions of oscillator-like algebras associated with Laguerre and Jacobi polynomials.

Keywords: orthogonal polynomials, oscillator-like algebras, deformed oscillator algebras.

1. Introduction

The usual harmonic oscillator annihilation, creation and the number operators are defined respectively as

$$\mathfrak{a}\Psi_n = \sqrt{n}\Psi_{n-1}, \qquad n \geq 1, \qquad \mathfrak{a}^\dagger\Psi_n = \sqrt{n+1}\Psi_{n+1}, \qquad n \geq 0, \qquad N = \mathfrak{a}^\dagger\mathfrak{a}, \tag{1.1}$$

and $\mathfrak{a}\Psi_0 = 0$, where $\{\Psi_n\}_{n=0}^{\infty}$ is an orthonormal basis of the harmonic oscillator Fock space. In this case

$$[\mathfrak{a}, \mathfrak{a}^\dagger] = I, \qquad [N, \mathfrak{a}] = -\mathfrak{a}, \qquad [N, \mathfrak{a}^\dagger] = \mathfrak{a}^\dagger, \qquad (\mathfrak{a}^\dagger)^\dagger = \mathfrak{a}, \qquad N^\dagger = N$$

and the algebra generated by $\{I, \mathfrak{a}, \mathfrak{a}^\dagger, N\}$ is the usual Weyl–Heisenberg algebra. We call this algebra $\mathfrak{A}_{\mathrm{WH}}$. It is well known that the dimension of this algebra is four.

Several generalizations and deformations of the algebra $\mathfrak{A}_{\mathrm{WH}}$ have been studied in the literature, see for example [1–4, 6, 9–11, 16]. In generalizing or deforming the algebra $\mathfrak{A}_{\mathrm{WH}}$ we are inclined to stay as close as to the commutation relations of the algebra $\mathfrak{A}_{\mathrm{WH}}$. In the following we shall provide conditions, in terms of the coefficients of some recurrence relations satisfied by the Fock basis of generalized oscillator-like algebras, for such algebras to be of the same dimension as the algebra $\mathfrak{A}_{\mathrm{WH}}$.

In particular, in the following we shall discuss the dimensions of generalized oscillator algebras presented in [4] and the dimensions of a modified version of an oscillator-like algebra presented in [1].

In a recent paper [13] we have considered the dimension of generalized oscillator algebras associated with orthogonal polynomials, on the real line, that are orthogonal with respect to a symmetric probability measure.

Let $\mathfrak{H} = L^2(\mathbb{R}, d\mu)$, where μ is a probability measure on $\mathbb{R}$ with finite moments

$$\mu_n = \int_{-\infty}^{\infty} x^n d\mu(x). \tag{1.2}$$

These moments uniquely define the real sequences $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$ and the system of orthogonal polynomials $\{\Psi_n(x)\}_{n=0}^{\infty}$ satisfying the recurrence relation [4]

$$x\Psi_n(x) = b_n\Psi_{n+1}(x) + a_n\Psi_n(x) + b_{n-1}\Psi_{n-1}(x), \qquad \Psi_0(x) = 1, \\ b_{-1} = 0, \qquad n = 0, 1, 2, \ldots. \tag{1.3}$$

The polynomials (normalized) $\{\Psi_n(x)\}_{n=0}^{\infty}$ form an orthonormal basis for a Fock space associated with a generalized oscillator algebra provided that b_n's and μ_n's are connected by a specific relation [4]. There are two cases associated with (1.3) [4, 7, 8]:

(i) $a_n = 0$, symmetric case,
(ii) $a_n \neq 0$, nonsymmetric case.

The primary aim of this article is to investigate the dimension of an oscillator-like algebra obeying the recurrence relation (1.3). We shall provide necessary and sufficient conditions, in terms of a_n and b_n of (1.3), for such an oscillator-like algebra to be of finite dimension. This result, in a manner, can be viewed as a dimensionwise classification for such algebras.

The rest of the article is organized as follows. In Subsection 2.1 we briefly discuss the symmetric case. In particular, we shall respond to the claims made in [7, 8] about the results of our earlier paper [13]. In Section 2.2 we discuss the nonsymmetric case and also comment on the results provided in [7, 8] about the oscillator algebra associated with the nonsymmetric case. Subsection 2.3 deals with oscillator-like algebras obeying the recurrence relation (1.3). In Section 3 we present the main results of this paper. That is, we present a necessary and sufficient condition on a_n and b_n of (1.3) for oscillator-like algebras to be of finite dimension. Some examples accommodating our claim are presented in Section 4. Section 5 ends paper with a conclusion.