QUANTUM FIELD THEORY APPLICATIONS OF HEUN TYPE FUNCTIONS

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After a brief introduction to Heun type functions we note that the actual solutions of the eigenvalue equation emerging in the calculation of the one-loop contribution to QCD from the Belavin–Polyakov–Schwarz–Tyupkin instanton and the similar calculation for a Dirac particle coupled to a scalar CP^1 model in two dimensions can be given in terms of confluent Heun equations in their original forms. These equations were previously modified to be solved by more elementary functions. We also show that polynomial solutions with discrete eigenvalues are impossible to find in the unmodified equations.

Keywords: Heun functions, quantum field theory, Dirac equation.

1. Introduction

Heun functions were introduced in 1889 by Karl Heun [1]. These functions are solutions of differential equations with four regular singularities. Any second-order differential equation with four regular singularities can be reduced to this form.

Although Heun equation was introduced in 1889, most theoretical physicists were not acquainted with this equation until 1990s. An important point in the history of this equation and its confluent forms is the "Centennial Workshop on Heun Equations: Theory and Applications, Sept. 3–8, 1989, Schloss Ringberg". The presentations at this workshop were printed in a book edited by A. Ronveaux [2]. After this meeting, this equation became more popular among theoretical physicists. After 1990s, it is encountered in increasingly many papers in theoretical physics. Work done before the year 2000 can be found in the book by Slavyanov and Lay [3]. For later applications, especially in general relativity, Hortaçsu gives a long list [4].

Note that Heun was not cited in the celebrated work of Teukolsky [5] when he wrote his Teukolsky master equations. These equations were shown to be reducible to a form of the Heun class later. Batic and Schmid [6] give references to people who tried to show whether Teukolsky master equations were the ones of the confluent forms of the Heun equation [7–9]. Then Batic et al., in the reference given above, showed that Teukolsky master equation could be transformed in any physically relevant D-type metric into a Heun form. In this paper we will try to identify one equation which was studied by 't Hooft in 1976, and show that in its original form, it is a confluent form of the Heun equation.

The Heun type solution is not specific for D-type metrics. Scalar and spinor wave equations, when written in the background of a Petrov I type instanton metric with two commuting Killing vectors and a Killing tensor also have Heun type solutions [10–12].

The eigenvalue equation emerging in the calculation of the one-loop contribution to QCD [13] from the Belavin–Polyakov–Schwarz–Tyupkin (BPST) instanton [14] and the calculation for a Dirac particle coupled to a scalar CP^1 model in two dimensions [15] were shown to yield hypergeometric functions, after these equations were modified slightly. Actually, the original equations give confluent Heun type solutions. These authors do not mention Heun type solutions in their papers.

We should note that it was the remarkable foresight of Gerard 't Hooft to devise a way to obtain a meaningful result from a complicated equation. If the equation was not modified, his result, which was of great importance in those days, would not have been obtained. Here, we will just show that the two unaltered equations are of confluent Heun type.

Polynomial solutions of confluent Heun equation can be obtained if the parameters of the equation satisfy certain conditions [16–20]. We will show that our quantum field theory examples do not satisfy these conditions and hence they do not have polynomial solutions.

2. Heun equation

The Heun equation is the most general second-order differential equation with four regular singular points. Usually these singular points are taken to be at zero, one, an arbitrary point d (other than zero and one) and at infinity. The general equation reads

$$\frac{d^2H_g}{dz^2}+\left[\frac{\gamma}{z}+\frac{\delta}{z-1}+\frac{\epsilon}{z-d}\right]\frac{dH_g}{dz}-\frac{\alpha\beta z-q}{z(z-1)(z-d)}H_g=0, \tag{1}$$

where H_g is a solution of Heun's general equation. We have the Fuchs relation between the constants given as $\alpha+\beta+1=\gamma+\delta+\epsilon$ which ensures regularity of the singularity at infinity. Here, q is the accessory parameter. The distinctive property of Heun function is not having a two-way recursion relation between the coefficients when one expands the solution as an infinite power series around one of its regular singular points. The recursion relation may contain at least three, sometimes more, coefficients, making it hard to extract the asymptotic properties from this expansion. Also, *no example has been given of a solution of Heun's equation expressed in the*