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THE SERIES PRODUCT FOR GAUSSIAN QUANTUM INPUT PROCESSES

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We present a theory for connecting quantum Markov components into a network with quantum input processes in a Gaussian state (including thermal and squeezed). One would expect on physical grounds that the connection rules should be independent of the state of the input to the network. To compute statistical properties, we use a version of Wicks' theorem involving fictitious vacuum fields (Fock space based representation of the fields) and while this aids computation, and gives a rigorous formulation, the various representations need not be unitarily equivalent. In particular, a naive application of the connection rules would lead to the wrong answer. We establish the correct interconnection rules, and show that while the quantum stochastic differential equations of motion display explicitly the covariances (thermal and squeezing parameters) of the Gaussian input fields we introduce the Wick–Stratonovich form which leads to a way of writing these equations that does not depend on these covariances and so corresponds to the universal equations written in terms of formal quantum input processes. We show that a wholly consistent theory of quantum open systems in series can be developed in this way, and as required physically, is universal and in particular representation-free.

Keywords: Gaussian Wick theorem, Wick–Stratonovich form, quantum Gaussian feedback networks.

1. Introduction

The quantum input-output theory has had an immense impact on quantum optics, and in recent years has extended to opto-mechanical systems and beyond. The prospect of routing the inputs through a network, or indeed using feedback has lead to a burgeoning field of quantum feedback control [1–5]. The development of a systems engineering approach to quantum technology has benefited from having a systematic framework in which traditional open quantum systems models can be combined according to physical connection architectures.

The initial work on how to cascade two quantum input-output systems can be traced back to Gardiner [6] and Carmichael [7]. More generally, the authors have introduced the theory of *Quantum Feedback Networks* (QFN) which generalizes this to include cascading, feedback, beam-splitting and general scattering of inputs, etc., [8, 9]. One of the basic constructs is the series product which gives the instantaneous feedforward limit of two components connected in series via quantum input processes: in fact, the systems need not necessarily be distinct and the series product generalizes cascading by allowing for feedback. The original work was done for input processes where the input fields where in the Fock vacuum field state. A generalization to squeezed fields and squeezing components has been given [10], however this was restricted to the case of linear coupling and dynamics: there it was shown that the resulting transform analysis could be applied in a completely consistent manner. More recent work has shown that nonclassical states for the input fields, such as shaped single-photon or multi-photon states, or cat states of coherent fields, may in principle be generated from signal models [11, 12]—that is, where a field in the Fock vacuum state was passed through an ancillary dynamical system (the signal generator) which is then to be cascaded to the desired system. Quantum feedback network (QFN) theory concerns the interconnection of open quantum systems. The interconnections are mediated by quantum fields in the input-output theory [8, 9, 13]. The idea is that an output from one node is fed back in as input to another (not necessarily distinct) node, the simplest case being the cascade connection (e.g. light exiting one cavity being directed into another). The components are specified by Markovian models determined by SLH parameters which describe the self-energy of the system and how the system interacts with the fields (via idealized Jaynes–Cummings type interactions and scattering).

Here we turn to the problem of the general class of Gaussian states for quantum fields. This includes thermal fields, and of course squeezed fields. In principle, these may be approximated as the output of a degenerate parametric amplifier (DPA) driven by vacuum input, see [13]. In a sense, we have that a singular DPA may serve is the appropriate signal generator to modify a vacuum field into a squeezed field before passing into a given network. We will exploit this in the paper, however, we will have to pay attention to the operator ordering problem when inserting these approximations into quantum dynamical equations of motion and input-output relations.

The programme turns out to be rather more involved than one might expect at first glance. It is always possible to represent a collection of d Gaussian fields using $2d$ vacuum fields (a Bogoliubov transformation!) and one might hope that the corresponding connection rules applied to the representation in terms of vacuum fields would agree with the intuitive rules one would desire. This turns out not to be the case, and the various feedback constraints cannot be naively applied to the representing fields: the reason is that the representations are a linear combination of creation and annihilation operators for the representing vacuum fields, and we have broken the Wick ordered form of the original equations.

If applied naively, the series product would predict a contribution to the global network model that depended on the covariance parameters of the state. From the physical point of view, this ought to be spurious. In comparison with classical