ON EQUI-TRANSMITTING MATRICES

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Equi-transmitting scattering matrices are studied. A complete description of such matrices up to order five is given. It is shown that the standard matching conditions matrix is essentially the only equi-transmitting matrix for orders 3 and 5. For orders 4 and 6, there exists other equi-transmitting ones but all such matrices have zero trace.

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1. Introduction

Quantum graphs—ordinary differential operators on metric graphs—is a rapidly developing area of modern mathematical physics. We refer to recent books [1, 11, 16] for references and overview. These models are widely used nowadays in different areas of applied mathematics: from nano-electronics to chemical physics. Mathematical studies of quantum graphs opened a new field of research with interesting spectral phenomena giving rise to development of new techniques and reconsidering old methods, originally developed for ordinary and partial differential equations on one side and discrete graphs on the other one. One of the central questions in the theory of quantum graphs is how good (or bad) these models reflect real-life systems. In the models the phase space is reduced to a collection of one-dimensional manifolds, while in the real-life systems the phase space can be a finite width neighbourhood of the union of manifolds. For example, numerous articles are devoted to understanding of the limiting procedure as the width turns to zero [4, 5, 10, 16]. The most difficult part is to understand which matching conditions at the vertices appear in the limit. These conditions not only make the differential operator self-adjoint (symmetric) but describe transition probabilities between the edges joined in a vertex.

A quantum graph can be seen as a triple consisting of a metric graph Γ, a differential expression L on the edges and certain matching conditions at the vertices. It is clear that to study matching conditions it is enough to consider star graphs, since matching conditions for general graphs should be introduced separately at each vertex. A widely used set of vertex conditions is the set of the so-called standard (Kirchhoff) matching conditions (SMC) on the functions at a vertex V

$$\begin{cases} u \text{ is continuous at } V, \\ \sum_{x_j \in V} \partial_n u(x_j) = 0, \end{cases} \tag{1.1}$$

where $\partial_n u$ denotes the outward derivative of u. Such matching conditions often appear in the limiting procedure described above. The corresponding vertex scattering matrix is [6, 9, 12, 13]

$$S_v = \begin{pmatrix} \frac{2}{v} - 1 & \frac{2}{v} & \frac{2}{v} & \cdots \\ \frac{2}{v} & \frac{2}{v} - 1 & \frac{2}{v} & \cdots \\ \frac{2}{v} & \frac{2}{v} & \frac{2}{v} - 1 & \\ \vdots & \vdots & & \ddots \end{pmatrix}, \tag{1.2}$$

where v is the valency, or degree, of the vertex—the number of edges joined together. In this particular case the vertex scattering matrix does not depend on the energy $E = k^2$ of scattered waves.

Standard matching conditions are widely used by quantum graph community for two reasons:

- in the case of degree two, $v = 2$, the scattering matrix degenerates to $S = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ meaning that the waves penetrate such a vertex without any reflection and without any change—these conditions do not introduce any interaction at the vertex and therefore sometimes are called *free conditions*;
- requirement that functions from the domain of the quadratic form are continuous at all vertices appears very natural, the corresponding self-adjoint operator is described by standard matching conditions (SMC) if the quadratic form does not contain any terms supported by the vertex.

But it was noted in [9] that such scattering matrix does not appear appropriate in the case of large degree $v \gg 1$, since the diagonal elements tend to -1 as $v \to \infty$, which means that a large portion of incoming waves are simply reflected back by such a vertex.

On the other hand, standard matching conditions (1.1) have another important property: the corresponding vertex scattering matrix is invariant under permutation of the edges—all nondiagonal entries are equal to $2/v$ and the diagonal one to $-1 + 2/v$. Our aim is to study all such matching conditions describing systems with permutation symmetry. The requirement that the entries in the vertex scattering matrix are equal is not very well justified from the point of view of quantum mechanics, where the transition probabilities are given by squares of absolute values. Therefore we shall describe here all matching conditions leading to vertex scattering matrices with modular-permutation symmetry

$$|s_{ij}| = |s_{kl}|, \qquad i \neq j, \ k \neq l, \tag{1.3}$$