# LIE SYMMETRIES AND CONSERVED QUANTITIES OF THE CONSTRAINT MECHANICAL SYSTEMS ON TIME SCALES

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We introduce a new method to study Lie symmetries and conserved quantities of constraint mechanical systems which include Lagrangian systems, nonconservative systems and nonholonomic systems on time scales $\mathbb{T}$. For the constraint mechanical systems on time scales, based on the transformation Lie group, we get a series of significant results including the variational principle of systems on time scales, the equations of motion, the determining equations, the structure equations, the restriction equations as well as the Lie theorems of the Lie symmetries of the systems on time scales. Furthermore, a set of new conserved quantities of the constraint mechanical systems on time scales are given. More significant is that this work unifies the theories of Lie symmetries of the two cases for the continuous and the discrete constraint mechanical systems by applying the time scales. And then taking the discrete ($\mathbb{T} = \mathbb{Z}$) nonholonomic system for example, we derive the corresponding discrete Lie symmetry theory. Finally, two examples are designed to illustrate these results.

**Keywords:** time scale, Lie symmetry, constraint mechanical system.

## 1. Introduction

The theory of time scales originated in 1988 with the work of Stefan Hilger [1] in order to unify various concepts from the theories of discrete and continuous dynamical systems, and to extend such theories to more general classes of dynamical systems. The time scales calculus theory is applicable to any field in which dynamic processes can be described with discrete or continuous models. The study of the calculus of variations in the context of time scales has its beginning in 2004 with the paper of Martin Bohner [2], with two excellent books dedicated to it [3, 4]. Since the pioneering paper [2], the classical results of the calculus of variations on continuous time ($\mathbb{T} = \mathbb{R}$) and discrete time ($\mathbb{T} = \mathbb{Z}$) have been unified and generalized to

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a time scale $\mathbb{T}$: Euler–Lagrange equations [5, 6], necessary optimality conditions for variational problems subject to isoperimetric constraints [7, 8], boundary value problems [9, 10], high-order delta derivatives [11–13], weak maximum principle for variable endpoints optimal control problems [14], an invariant group of parameter transformations [15]. The more general theory of the calculus of variations on time scales seems to be useful in applications to economics [16].

It is significant to study the symmetries and conserved quantities of dynamical systems not only in mathematics but also in physics. And the development of modern physics such as quantum mechanics, quantum field theory, nuclear physics, and spatially inhomogeneous nonlinearities shows that the principle of symmetry has become the most important principle of exploring the laws of motion of microparticles. There are two main symmetry methods used to seek the conserved quantities of a dynamical system: the Noether symmetry and the Lie symmetry [17, 18]. The Noether method is making good progress, such as Herglotz variational problems [19], problems with isoperimetric constraints [20], problems with delay arguments [21] and problems with noninteger order derivatives [22]. The Lie method has been approved to be a powerful tool to solve differential equations, to study constraint mechanical systems, to discuss controllable dynamical systems, to investigate mechanico-electrical systems and to establish properties of their solution spaces. The theories of Lie symmetry have been described in many papers [23–27]. The Lie group theory has also been applied to discrete equations, such as differential-difference equations, discrete dynamical systems and discrete mechanico-electrical systems [28, 29]. We have seen that the theories of Lie symmetry of the continuous and the discrete systems have been deeply studied and they are studied in two different directions. Here we introduce the concept of time scale which unifies the theories of Lie symmetries of the two cases for the continuous ($\mathbb{T} = \mathbb{R}$) and the discrete ($\mathbb{T} = \mathbb{Z}$) constraint mechanical systems.

This paper systematically studies the conserved quantities and Lie symmetries for constraint mechanical systems on an arbitrary time scale $\mathbb{T}$ which unifies and extends the previous formulations of Lie's method in the discrete-time and continuous domains (cf. [23–29] and references therein). By defining the isochronous variation on time scales, we study the exchanging relationships between the isochronous variation and the delta derivatives as well as the relationships between the isochronous variation and the total variation on time scales. On the basis of the invariance theory of differential equations of motion under the infinitesimal transformations, we study the Lie symmetry of the Lagrangian system. By virtue of the Hamilton principle, we construct the equations of motion of nonconservative systems on time scales, and then we give the method of solution of the Lie symmetries of nonconservative systems on time scales. Furthermore, using the d'Alembert–Lagrange principle on time scales we derive the equations of motion of nonholonomic systems of Chetaev type. Then, by presenting the Lie symmetrical determining equations, the structure equations and the constrained restriction equations of the constraint mechanical systems on time scales $\mathbb{T}$, we get a new type of Noether conserved quantity which only depends on the variables $t, q, q^{\sigma}, q^{\Delta}$. For the classical discrete time ($\mathbb{T} = \mathbb{Z}$)