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SUPERSYMMETRY OF THE MORSE OSCILLATOR

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While dealing in [1] with the supersymmetry of a tridiagonal Hamiltonian H, we have proved that its partner Hamiltonian $H^{(+)}$ also has a tridiagonal matrix representation in the same basis and that the polynomials associated with the eigenstates expansion of $H^{(+)}$ are precisely the kernel polynomials of those associated with H. This formalism is here applied to the case of the Morse oscillator which may have a finite discrete energy spectrum in addition to a continuous one. This completes the treatment of tridiagonal Hamiltonians with pure continuous energy spectrum, a pure discrete one, or a spectrum of mixed discrete and continous parts.

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1. Introduction

In a previous work [1], we have been concerned with a positive semi-definite Hamiltonian H, whose matrix representation in a chosen basis is tridiagonal. Precisely, we have defined a forward-shift operator A and its adjoint, the backward-shift operator $A^\dagger$, by specifying how they act on each vector basis. It turns out that these operators play a central role in our treatment. Their matrix representations have been derived by demanding that the given Hamiltonian has the form $A^\dagger A$. We proved that $A^\dagger A$ also has a tridiagonal matrix representation in the chosen basis and we established explicit formulae connecting the parameters defining A and the matrix elements of H. These parameters are closely related to the set of polynomials associated with the eigenstates expansion of H in the chosen basis. Writing the supersymmetric partner Hamiltonian $H^{(+)}$ as $AA^\dagger$, we showed that it also has a tridiagonal matrix representation in the same basis. We then have established that polynomials associated with $H^{(+)}$ are precisely the kernel polynomials of those associated with H. The applications of these results have been illustrated to two well-know Hamiltonians, namely, the free particle Hamiltonian whose energy spectrum is purely continuous and the harmonic oscillator Hamiltonian whose energy spectrum is purely discrete. We confirmed that our treatment reproduced previously established results for these two systems.

In this paper, our aim is to test our above formalism on a system whose spectrum is both discrete and continous. The prototypical system having this property is the Morse oscillator whose supersymmetry formalism has been investigated and settled a long time ago. Another advantage for choosing this oscillator lies in knowing an explicit basis set that renders its Hamiltonian tridiagonal.

The Morse potential is usually written as $V(x) = V_0(e^{-2\alpha x} - 2e^{-\alpha x})$ where V_0 and α are given parameters related to the properties of the physical system that this potential is attempting to model. In particular, V_0 controls the depth of the potential while α controls its width. For the purpose of this paper, we assume that the values of V_0 and α are such that the potential possesses one or more bound states.

The paper is organized as follows. In Section 2, we review briefly the formalism developed to treat the supersymmetry of tridiagonal Hamiltonians. In Section 3, we apply this formalism to the specific case of the Morse oscillator. We then identify the basis set that renders the Hamiltonian tridiagonal. In the process of finding the representation of the energy eigenvectors in the same basis, we solve the resulting three-term recursion relation satisfied by a set of orthogonal polynomials. The solution includes identification of the discrete spectrum and the form of the completeness relation satisfied by the orthogonal polynomials. We then find the matrix representation of the supersymmetric partner Hamiltonian and show, in an analogous manner, how to completely characterize its properties. In Section 4, we discuss the obtained results and derive additional ones to suggest that our formalism is a viable tool in the study of supersymmetry.

2. Summary of results on supersymmetry of tridiagonal Hamiltonians

In this section we summarize the results of our previous work on supersymmetry of tridiagonal Hamiltonians [1].

We assume that the matrix representation of the given Hamiltonian H in a complete orthonormal basis $|\phi_n\rangle$, $n = 0, 1, 2, \ldots,$ is tridiagonal. That is

$$\langle \phi_n \mid H \mid \phi_m \rangle = b_{n-1}\delta_{n,m+1} + a_n\delta_{n,m} + b_n\delta_{n,m-1}. \tag{2.1}$$

We now define the forward-shift operator A by its action on the basis $|\phi_n\rangle$ as follows

$$A\,|\phi_n\rangle = c_n\,|\phi_n\rangle + d_n\,|\phi_n\rangle \tag{2.2}$$

for every $n = 1, 2, \ldots$. For $n = 0$, we state that $d_0 = 0$. Furthermore, we require from the adjoint operator $A^\dagger$ to act on the ket vectors $|\phi_n\rangle$ as

$$A^\dagger\,|\phi_n\rangle = c_n\,|\phi_n\rangle + d_{n+1}\,|\phi_n\rangle\,, \qquad n = 0, 1, 2, 3, \ldots. \tag{2.3}$$