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ORDER REDUCTION, PROJECTABILITY AND CONSTRAINTS OF SECOND-ORDER FIELD THEORIES AND HIGHER-ORDER MECHANICS

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The projectability of Poincaré–Cartan forms in a third-order jet bundle $J^3\pi$ onto a lower-order jet bundle is a consequence of the degenerate character of the corresponding Lagrangian. This fact is analyzed using the constraint algorithm for the associated Euler–Lagrange equations in $J^3\pi$. The results are applied to study the Hilbert Lagrangian for the Einstein equations (in vacuum) from a multisymplectic point of view. Thus we show how these equations are a consequence of the application of the constraint algorithm to the geometric field equations, meanwhile the other constraints are related with the fact that this second-order theory is equivalent to a first-order theory. Furthermore, the case of higher-order mechanics is also studied as a particular situation.

Keywords: 2nd-order Lagrangian field theories, higher-order mechanics, Poincaré–Cartan form, Einstein-Hilbert action.

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1. Introduction

There are some models in classical field theories where, as a consequence of the singularity of the Lagrangian, the order of the Euler–Lagrange equations is lower than expected. A geometrical way of understanding this problem is considering the projectability of the higher-order Poincaré–Cartan form onto lower-order jet bundles [3, 13, 14, 16, 17]. We review the conditions for this projectability and study their consequences using the constraint algorithm for the field equations of second-order (singular) field theories, thus enlarging the results stated in previous papers [3, 9, 14, 16, 17]. This constitutes the main result of the paper and it is stated in Theorem 2.

In this paper we restrict our study to second-order field theories in order to avoid some kinds of problems involving the ambiguity in the definition of the Poincaré–Cartan form in a higher-order jet bundle, the nonuniqueness of the construction of the Legendre map associated with a higher-order Lagrangian and the choice of the multimomentum phase space for the Hamiltonian formalism [1, 8, 10, 12, 13]. As it is well known, for the second-order case, all the Poincaré–Cartan forms are proved to be equivalent and the Legendre map and the Hamiltonian multimomentum phase space can be unambiguously defined [15, 18, 19].

As a relevant example, the case of the Hilbert Lagrangian for the Einstein equations with no matter sources is analyzed. In particular, we show how these equations are obtained as constraints appearing as a consequence of the application of the constraint algorithm to the geometric field equations which are stated in the corresponding third-order jet bundle. The other constraints arising in the algorithm are of geometrical nature. They are related with the fact that we are working with some unnecessary degrees of freedom, because we are using a third-order jet bundle to describe a second-order theory that, as a consequence of the projectibility of the Poincaré–Cartan form, is really equivalent to a first-order theory [17]. In addition, this study constitutes a new approach to a multisymplectic formulation of the Lagrangian formalism for this model, which is different from other previous attemps to this subject [20].

Finally, this analysis is done for the case of higher-order mechanics which, as it is well known, can be considered as a particular case of higher-order field theories. Here we consider dynamical systems of any order, since the mentioned above ambiguities about the construction of the Poincaré–Cartan form and the Legendre map do not occur in higher-order tangent bundles.

All the manifolds are real, second countable and C^∞. The maps and the structures are C^∞. The sum over repeated indices is understood. In order to use coordinate expressions, remember that a multi-index I is an element of $\mathbb{Z}^m$ where every component is positive, the ith position of the multi-index is denoted $I(i)$, and $|I| = \sum_{i=1}^{m} I(i)$ is the length of the multi-index. An expression as $|I| = k$ means that the expression is taken for every multi-index of length k. Furthermore, the element $1_i \in \mathbb{Z}^m$ is defined as $1_i(j) = \delta_i^j$. Finally, $n(ij)$ is a combinatorial factor defined as $n(ij) = 1$ for $i = j$, and $n(ij) = 2$ for $i \neq j$.

2. Order reduction and projectability of the Poincaré–Cartan form

Let M be an m-dimensional manifold and $\pi\colon E \to M$ a fiber bundle over M with $\dim E = m + n$ (the *configuration bundle* of a classical field theory). The k-jet manifold of π is denoted $J^k\pi$ and is endowed with the natural projections $\pi_s^k\colon J^k\pi \to J^s\pi$, $\pi^k\colon J^k\pi \to E$, $\overline{\pi}^k\colon J^k\pi \to M$; for $k > s \geq 0$. Then, a section $\psi\colon M \to J^k\pi$ of $\bar{\pi}^k$ is *holonomic* if $j^k(\pi^k \circ \psi) = \psi$; that is, ψ is the kth prolongation of a section $\phi = \pi^k \circ \psi\colon M \to E$.

Remember that a form $\omega \in \Omega^s(E)$ is said to be π*-semibasic* if $i(X)\omega = 0$, and π*-basic* or π*-projectable* if $i(X)\omega = 0$ and $\mathrm{L}(X)\omega = 0$, for every π-vertical vector field $X \in \mathfrak{X}^V(\pi)$ (here, the symbols i and L denote the inner contraction and the Lie derivative, respectively). As a consequence of Cartan's formula, $\mathrm{L}(X)\omega = i(X)\mathrm{d}\omega + \mathrm{d}\,i(X)\omega$, a form $\omega \in \Omega^n(E)$ is π-basic if, and only if, ω and $\mathrm{d}\omega$ are π-semibasic.