NONCOMMUTATIVE VALUATION OF OPTIONS

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The aim of this note is to show that the classical results in finance theory for pricing of derivatives, given by making use of the *replication principle*, can be extended to the noncommutative world. We believe that this could be of interest in quantum probability. The main result called the *First fundamental theorem of asset pricing*, states that a noncommutative stock market admits no-arbitrage if and only if it admits a noncommutative equivalent martingale probability.

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1. Introduction

The aim of this article is to provide a noncommutative setting for pricing of derivatives. Indeed, the main idea is to provide the precise formulation and hypotheses in the noncommutative world in order to be able to prove the basic results in option pricing by applying the so-called *replication principle*. In particular, we prove the *First fundamental theorem of asset pricing* (see Theorem 1), stating that a noncommutative stock market admits no-arbitrage if and only if it admits a noncommutative equivalent martingale probability. There are several obstacles to tackle which do not appear in the classical situation, such as the problems with the notion of conditional expectation in the noncommutative setting (see Example 4), which in turn implies that the definition of martingale is not very clear. The article can be thus regarded as a proposal of a collection of definitions, which work in the noncommutative world and thus lay a rather simple theoretical framework, that allow us to extend the classical results in finance theory of pricing of derivatives. Of course, many of the steps we shall follow are greatly inspired by the classical theory, which we more or less try to follow.

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Our point of view of noncommutative probability theory is mainly algebraic (see [12] and also the very interesting [21]), and we usually refrain from imposing analytic conditions, such as being a C^*-algebra or a W^*-algebra, but simpler algebraic hypotheses which extract the only required properties in order to prove our theorems. In particular, this theory can be used to understand examples that appeared in the literature, as the toy quantum binomial model in [7]. We remark however that our point of view is in principle different from [5, 6]. It is also different from the work appearing in [19] or [2, 3], for we do not use any specific physical model to describe the price processes or the trading strategies.

Our main motivation is given by noncommutative probability spaces coming from quantum theory (see Example 2). This becomes apparent in case one considers markets modelled using quantum computations, when both the price processes and the trading strategies should follow quantum principles, as the toy model in [7]. A particular field of (future) interest could be to develop quantum versions of *High frequency trading* (HFT), implemented using quantum computers, even though the (classical) theory is in principle different.

2. Noncommutative probability spaces

2.1. Basic definitions

We recall now the basic definitions of noncommutative probability theory given by T. Tao in [21], Section 2.5. In order to do so, we introduce first some basic notion of the theory of involutive algebras (see [17], Chapter 9, and references therein for further details).

We recall that a $*$-*vector space* consists of a vector V over the complex numbers $\mathbb{C}$ together with a sesquilinear map $(-)^* : V \to V$ (i.e. $(v + cw)^* = v^* + \bar{c}w^*$, for all $v, w \in V$ and $c \in \mathbb{C}$, where $\bar{c}$ denotes the complex conjugate of c) which is an involution, i.e. $(v^*)^* = v$, for all $v \in V$. To reduce notation we will typically denote a $*$-vector space $(V, (-)^*)$ by its underlying vector space V. Note that $\mathbb{C}$ is a $*$-vector space with the involution given by complex conjugation. Given two $*$-vector spaces V and W, a *morphism* from V to W (also called a $*$-*linear map*) is a $\mathbb{C}$-linear map $f : V \to W$ between the underlying vector spaces that commutes with the involutions, i.e. $f(v)^* = f(v^*)$, for all $v \in V$.

A $*$-*algebra* (or *involutive algebra*) is a $*$-vector space A together with an associative $\mathbb{C}$-algebra structure on the underlying vector space of A such that $(ab)^* = b^*a^*$, for all $a, b \in A$. We further say that it is a *unitary* $*$-algebra if the underlying associative algebra has a unit 1_A. Note that the uniqueness of the unit implies that $1_A^* = 1_A$. A *morphism* of (resp. unitary) $*$-algebras is a $*$-linear map which is also a morphism of the underlying (resp. unitary) associative algebras. Given a set $S \subseteq A$ satisfying that $S^* \subseteq S$, we denote by $\mathscr{C}(S)$ the space formed by all elements $x \in A$ which commute with all elements of S. It is usually called the *centralizer* (or *commutant*) of S. Note that the condition $S^* \subseteq S$ implies that $\mathscr{C}(S)$ is a sub-$*$-algebra of A.