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Error estimates for Gaussian beam methods applied to symmetric strictly hyperbolic systems

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HIGHLIGHTS

- A complete Gaussian-Beam construction is presented for a class of linear symmetric hyperbolic systems with highly oscillatory initial data, including both strictly and non-strictly hyperbolic systems as long as they are diagonalizable.
- The evolution equations for each Gaussian beam component are derived.
- Independent of dimension and presence of caustics an optimal error estimate between the exact solution and the first order Gaussian beam superposition in terms of the high frequency parameter is obtained.
- Further applications to some non-strictly hyperbolic systems including both the acoustic equation and the system of Maxwell equations are discussed.

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ABSTRACT

In this work we construct Gaussian beam approximations to solutions of linear symmetric hyperbolic systems with highly oscillatory initial data, including both strictly and non-strictly hyperbolic systems as long as they are diagonalizable. The evolution equations for each Gaussian beam component are derived. Under some regularity assumptions of the data we obtain an error estimate between the exact solution and the first order Gaussian beam superposition in terms of the high frequency parameter ε^{-1}. The main result is that the relative local error measured in energy norm in the beam approximation decays as $\varepsilon^{\frac{1}{2}}$ independent of dimension and presence of caustics, for first order beams. This result is shown to be valid when the gradient of the initial phase is bounded away from zero. Applications to some non-strictly hyperbolic systems including both the acoustic equation and the system of Maxwell equations are discussed.

1. Introduction

In this article we are interested in the accuracy of Gaussian beam approximations to solutions of the hyperbolic system,

$$A(x)\frac{\partial u}{\partial t} + \sum_{j=1}^{n} D^j \frac{\partial u}{\partial x_j} = 0, \tag{1.1}$$

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subject to highly oscillatory initial data,

$$u(0,x) = B_0(x)e^{iS_0(x)/\varepsilon}, \tag{1.2}$$

where $x \in \mathbb{R}^n$, $S_0(x)$ is a scalar smooth function, $B_0 : \mathbb{R}^n \to \mathbb{C}^m$ is a smooth vector function, compactly supported in $K_0 \subset \mathbb{R}^n$, $A(x)$ is $m \times m$ symmetric positive definite matrix, and D^j are $m \times m$ symmetric constant coefficient matrices, $j = 1, \ldots, n$.

Symmetric systems first considered by K.O. Friedrichs can be used to describe a wide variety of physical processes. Indeed, with proper choices of A and D^j one may find in (1.1) Maxwell's equations, the elastic wave equations, and acoustic wave equations as discussed in [1]. In several areas of continuum physics including acoustic waves, the wave length parameter ϵ is often very small compared to the scale of computational domain, hence one has to solve high frequency wave propagation problems, and the research in this field can give some insight in the study of some significant physical systems. The symmetry of the hyperbolic system ensures the existence of the orthogonal basis in $\mathbb{R}^n$ formed by the associated eigenvectors, and this spectral decomposition is useful in our construction of high frequency approximate solutions. Indeed many hyperbolic equations or systems can be reduced to symmetric hyperbolic form. A typical example is a scalar hyperbolic second or higher order equation.

It is well-known that high frequency wave propagation problems create severe numerical challenges that make direct simulations unfeasible, particularly in multidimensional settings. High frequency asymptotic models, such as geometrical optics, can be found in some classical literature (see [1,2]). A main drawback of geometrical optics is that the model breaks down at caustics, where rays concentrate and the predicted amplitude becomes unbounded, therefore unphysical. As an alternative one can use the phase space based level set method, to compute multi-valued phases beyond caustics, we refer to [3–6] for the early development of this method, and [7] for a review of the level set framework for computational high frequency wave propagation. The density transport near the level set manifold produces bounded position densities everywhere except at caustics [8,9].

Gaussian beams, as another high frequency asymptotic model, are closely related to geometric optics, yet valid at caustics. The solution is concentrated near a single ray of geometric optics. In Gaussian beams, the phase function is real valued along the central ray, its imaginary part is chosen so that the solution decays exponentially away from the central ray, maintaining a Gaussian shaped profile. More general high frequency solutions can be described by superposition of Gaussian beams. In this paper we are going to use the Gaussian beam approach. This approach has gained considerable attention in recent years from both computational and theoretical points of view. A general overview of the history and the latest development of this method are given in the introduction to [10]. We remark that the phase space based level set method when combined with the Gaussian beam framework, as developed in [11–13], can both handle the crossing of Hamiltonian trajectories and yield bounded amplitudes at caustics. A systematic Gaussian beam construction using a level set formulation is presented in [14,15] together with rigorous error estimates.

Another related approach is the frozen Gaussian approximation, or the Herman–Kluk formula discovered by several authors in the chemical-physics literature in the eighties. This approach with superposition of beams in phase space is closely related to the Fourier-Integral Operator (FIO) with complex phases. The mathematical analysis of the Herman–Kluk was given only recently; see [16,17] for the semiclassical approximation of the Schrödinger equation, and [18] for the frozen Gaussian approximation to linear strictly hyperbolic systems.

In this paper we formulate a Gaussian beam superposition in physical space for symmetric hyperbolic systems. Though the Gaussian beam construction is standard after [19], we show how the Gaussian beam works in the case of systems, which requires additional care compared to scalar equations. We note, in particular, that a higher order amplitude term, $\varepsilon v_1^\top$, is needed even for first order beams to account for the spatial variations in the eigendirections. We mainly study the accuracy in terms of the high frequency parameter ε of Gaussian beams. Several such error estimates have been derived in recent years for problems modeled by several different types of PDEs: for the initial data [20], for scalar hyperbolic equations and the Schrödinger equation [10,14,15], for the acoustic wave equation with superpositions in phase space [21], for the Helmholtz equation with a singular source [22], and for the Schrödinger equation with periodic potentials [23]. The general result is that the error between the exact solution and the Gaussian beam approximation decays as $\varepsilon^{N/2}$ for Nth order beams in the appropriate Sobolev norm. For phase space based Gaussian beams with frozen Gaussians, the integral approximation decays as ε^N for Nth order beams; see [16,17,24]. We note that in the frozen Gaussian beam approximation, the extra order of accuracy is found from a symbolic calculus for FIO with complex quadratic phases. But it is no longer a superposition of asymptotic solutions, though the superposition over phase space is still an asymptotic solution. While the Gaussian beam methods benefit mainly from the asymptotic accuracy of each individual beams, therefore it is computationally more feasible. We should also point out that Gaussian beam superposition in physical space works well for highly oscillatory data of the WKB type. For non-oscillatory initial data, one can simply apply a direct discretization method to solve (1.1). In contrast, in computing the semi-classical limit of the Schrödinger equation, one would have to handle oscillatory wave fields even for non-oscillatory initial data. In such case, one may carry out Gaussian beam superpositions in phase space, we refer to [12] for related strategies for using Gaussian wave packets and corresponding beam superpositions in phase space.

The analysis of Gaussian beam superpositions for hyperbolic systems presents a few new challenges compared to the scalar wave equations previously studied in [10,14]. First, it must be clarified how beams are propagated along each wave field through some field decomposition; a specific hyperbolic system was studied in [25], where the authors investigate the stationary in time wave field that results from steady air flow over topography. Second, the distinction of the eigenvalues of the dispersion matrix is assumed to allow for a correction in the amplitude with uniform estimates. This is similar to the