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# Sixth order solitary wave equations

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## HIGHLIGHTS

- Sixth order equations.
- Electrical solitons.
- Hirota bilinear operator.
- Sine-Gordon equation.

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ABSTRACT

In this work, we present higher order solitary wave equations, in particular sixth order. We show how these equations can be derived using fundamental physics laws, such as the Ohm's law. We use the Taylor series expansion and in some cases the Hirota's bilinear operator to obtain these model equations. The sixth order solitary wave equations model different physical problems such as problems in the electrical domain and the propagation of dispersive water waves.



## 1. Introduction

The solitary wave (localized wave with permanent character) was first observed by John Scott Russell on the surface of a shallow water layer (a canal near Edinburgh) and the theory of the solitary waves was developed by Boussinesq [1,2] to explain his observations. The work of Boussinesq introduced a new paradigm in which the existence of permanent waves in nonlinear systems is the result of the balance between nonlinearity and dispersion. Under the assumption of slow evolution in the frame moving with the center of the solitary wave, Boussinesq's equation can be reduced to the famous Korteweg and de Vries equation (KdV) for which Zabusky and Kruskal [3] discovered numerically wave solutions with particle-like behavior which they called *solitons*. Later on, the Boussinesq equation was shown to apply to the continuous limit of atomic lattices [4] (see, e.g., [5]) and to the flexural deformations of elastic rods. Solitons on elastic rods have been studied for different models and by different techniques (see, e.g. [6], and the monograph [7]). For the more fundamental information about the connection of solitary wave problems to homoclinic bifurcation, the reader is referred to the review [8].

Solitons have been studied intensively for the last 60 years now and theory and experimental results have been established. It has been proven that solitons exist only in integrable systems, therefore there is a great need of finding such systems since they are very rare to find. The work in this paper was motivated by the work of Christov [5] where a chain of points of equal masses, connected to each other through springs, was considered. After applying Newton's law for the mass point of number $n$, a discrete model was derived and using the continuum limit approximation the model equation was obtained.

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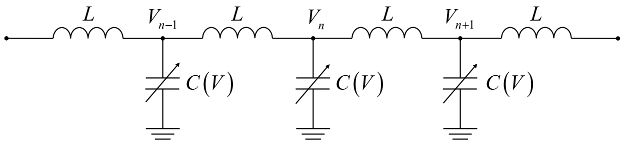


**Fig. 1.** A nonlinear transmission line.

In the current paper, problems arising from various areas of physical applications were considered. We begin by redefining the notion of nonlinear transmission line (NLTL) and the application of Ohm's law. The problem is rendered to a nonlinear Toda lattice where the force between masses is replaced by the voltage at sections of the NLTL, [9]. Then, after applying the Taylor series expansion, we end up to a sixth order Boussinesq type equation. Moreover, if the Boussinesq approximation is applied on the nonlinear term of the new model equation, we can show that the model is comparable to the equation proposed in [10]. The next problem considered, originates from the work of Fermi, Pasta and Ulam [11] and the work of Kruskal and Zabusky [12,13]. Here, a one dimensional lattice consisting of identical masses joined with springs was considered and Hooke's law is applied to evaluate the force exerted by extending or compressing the springs. The Taylor approximation is applied to produce a sixth order model equation. Finally, the Hirota bilinear operator is used on the Boussinesq equation to obtain a new sixth order equation, which is also comparable to the equation proposed in [10].

## 2. Sixth order equation using Ohm's law

In this work, our aim is to study solitons that occur in electrical transmission lines; these are called electrical solitons, [9,14,15]. In general, a transmission line is made of two wires used to transmit an electronic signal. When an electronic wave or pulse propagates along the transmission line its behavior depends on the nature of the wave along and the type of wires, as well as the interaction between the wires. In the current paper, we consider a transmission line which is constructed by an inductance per unit length $L$ and a capacitance per unit length $C$ between the wires, see Fig. 1. In the case where the capacitance is a linear function of the applied voltage $V$ the transmission line is called linear where as in the case where the capacitance is a nonlinear function of the voltage the transmission line is called nonlinear.

Ohm's law for an inductor is given by

$$V = L\frac{dI}{dt}, \tag{1}$$

where $V$ is the voltage, $L$ the inductance and $\frac{dI}{dt}$ is the instantaneous rate of current change. The nodal equations are

$$V_{n-1} - V_n = L\frac{dI_n}{dt}, \tag{2}$$

$$V_n - V_{n+1} = L\frac{dI_{n+1}}{dt}, \tag{3}$$

$$I_n - I_{n+1} = \frac{dQ_n(V)}{dt}, \tag{4}$$

where $Q_n(V)$ is the charge on the nth capacitor. If we differentiate Eq. (4) with respect to $t$ and make use of (1) we have

$$\frac{d^2Q_n(V)}{dt^2} = \frac{1}{L}\left[V_{n-1} - 2V_n + V_{n+1}\right]. \tag{5}$$

One may choose

$$Q_n(V) = Q_o \ln\left[1 + \frac{V_n}{F_o}\right], \tag{6}$$

and, Eq. (5) is rendered to

$$Q_oL\frac{d^2}{dt^2}\ln\left[1 + \frac{V_n}{F_o}\right] = \left[V_{n-1} - 2V_n + V_{n+1}\right], \tag{7}$$

where $F_o$ is Faraday's constant. Eq. (7) is basically a nonlinear Toda lattice where the force between masses is replaced by the voltage between sections of the nonlinear transmission line. Following the work of Christov [5], where a quadratic potential based on the three-point difference was used, that yields a linear term proportional to the five-point difference

$$V_{n-2} - 4V_{n-1} + 6V_n - 4V_{n+1} + V_{n+2}. \tag{8}$$

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